THE SEVENTEENTH ANNUAL MEETING OF THE AMERICAN MATHEMATICAL SOCIETY.

THE seventeenth annual meeting of the Society was held in New York City on Wednesday and Thursday, December 28-29, 1910, extending through the usual morning and afternoon sessions on each day. The attendance included the following sixty-three members of the Society:

Mr. F. W. Beal, Professor W. J. Berry, Professor G. D. Birkhoff, Professor Maxime Bôcher, Professor Joseph Bowden, Dr. Emily M. Coddington, Professor A. B. Coble, Professor F. N. Cole, Dr. J. R. Conner, Dr. G. M. Conwell, Professor J. L. Coolidge, Professor E. W. Davis, Professor F. F. Decker, Dr. L. S. Dederick, Professor W. P. Durfee, Professor L. P. Eisenhart, Professor J. C. Fields, Professor H. B. Fine, Professor W. B. Fite, Professor A. S. Gale, Professor O. E. Glenn, Professor C. C. Grove, Professor C. O. Gunther, Professor E. L. Hancock, Professor J. G. Hardy, Professor C. N. Haskins, Professor H. E. Hawkes, Dr. W. A. Hurwitz, Professor L. A. Howland, Mr. S. A. Joffe, Professor Edward Kasner, Professor C. J. Keyser, Dr. D. D. Leib, Dr. N. J. Lennes, Mr. Joseph Lipke, Professor W. R. Longley, Mr. A. R. Maxson, Professor Helen A. Merrill, Dr. H. H. Mitchell, Professor C. L. E. Moore, Professor C. .N Moore, Professor Frank Morley, Professor Richard Morris, Professor G. D. Olds, Dr. H. B. Phillips, Dr. H. W. Reddick, Professor R. G. D. Richardson, Miss S. F. Richardson, Mr. L. P. Siceloff, Mr. C. G. Simpson, Dr. Clara E. Smith, Mr. F. H. Smith, Professor P. F. Smith, Mr. W. M. Smith, Dr. W. M. Strong, Dr. M. O. Tripp, Professor C. B. Upton, Professor Oswald Veblen, Mr. H. E. Webb, Professor H. S. White, Mr. E. E. Whitford, Professor E. B. Wilson, Professor T. W. D. Worthen.

President Maxime Bôcher took the chair at the opening session, yielding it later to Professor Kasner, Ex-President Osgood, and at the morning session on Thursday to the President elect, Professor H. B. Fine. The Council announced the election of the following persons to membership in the Society: Professor Percy Hodge, Columbia University; Mr. C. G. P. Kuschke, University of California; Professor Marion B. White, University of Kansas. Eight applications for membership were received.

On Wednesday evening thirty of the members gathered at the annual dinner, which has for many years been a pleasant adjunct to the regular sessions.

It was decided to hold the summer meeting in 1911 at Vassar College. Professors H. S. White, P. F. Smith, and the Secretary were appointed a committee to make the necessary arrangements. The summer meeting of 1912 will be held at the University of Pennsylvania.

The report of the Treasurer, Auditing Committee, and Librarian are published in the Annual Register. The membership of the Society has increased during the past year from 618 to 641, including at present 60 life members. The total attendance of members at all meetings was 317. The number of papers presented was 145. In the annual election 202 votes were cast. The Society's library shows the normal yearly increase, the total number of books on the shelves being now 3,508. The Treasurer's report shows a balance of $8,124.53, a slight increase over last year, although the subventions received from various universities for the support of the *Transactions* have been discontinued, and the Society has paid a first installment of 1,000 francs of its contribution toward the publication of the collected works of Euler. The income from sales of the Society's publications during the year was $1,660.21. The life membership fund now amounts to $3,901.83.

During the past sixteen years the Society has expended for printing the BULLETIN, *Transactions*, and other publications $42,473.57. The total returns from publications have been in the same period $15,375.20. The university subventions have amounted to $8,100. Editorial and administrative expenses have been $11,779.49. The stock of publications on hand may be fairly valued at about $10,000.

At the annual election, which closed on Thursday morning, the following officers and other members of the Council were chosen:

President,	Professor H. B. FINE.
Vice-Presidents,	Professor G. A. BLISS, Professor W. E. STORY.
Secretary,	Professor F. N. COLE.
Treasurer,	Professor J. H. TANNER.
Librarian,	Professor D. E. SMITH.

Committee of Publication,
Professor F. N. COLE,
Professor E. W. BROWN,
Professor VIRGIL SNYDER.

Members of the Council to serve until December, 1913,

Professor H. F. BLICHFELDT, Professor C. J. KEYSER,
Professor J. L. COOLIDGE, Professor J. W. YOUNG.

The following papers were read at this meeting:

(1) Mr. S. CHAPMAN: "A note on the theory of summable integrals."

(2) Dr. H. H. MITCHELL: "Concerning a rotation group in six-space."

(3) Professor VIRGIL SNYDER: "An application of a (1–2) quaternary correspondence."

(4) Rev. A. S. HAWKESWORTH: "Three new dimension theorems."

(5) Professor W. R. LONGLEY: "Singular points on the discriminant locus of an ordinary differential equation."

(6) Professor R. G. D. RICHARDSON: "Theorems of oscillation for two self-adjoint linear differential equations of the second order with two parameters."

(7) President MAXIME BÔCHER: "A simple proof of a fundamental theorem in the theory of integral equations."

(8) Professor J. L. COOLIDGE: "The metrical aspect of the line-sphere transformation."

(9) Dr. E. J. MILES: "The absolute minimum of a definite integral in a special field."

(10) Professor J. C. FIELDS: "A method of proving certain theorems relating to rational functions which are adjoint to an algebraic equation for a given value of the independent variable."

(11) Professor F. F. DECKER: "Concerning the order of a restricted system of equations."

(12) Professor PAUL SAUREL: "On the classification of crystals."

(13) Professor A. B. COBLE: "An application of Moore's cross ratio group to the solution of the sextic equation."

(14) Professor A. B. COBLE: "The cubic surface and plane six-point."

(15) Professor C. L. E. MOORE: "Conjugate directions on a hypersurface in S_4 and some allied curves."

(16) Professor W. H. BATES: "An application of symbolic methods to the treatment of mean curvature in hyperspace."

(17) Dr. H. B. PHILLIPS: "The Galois theory of sets of multipartite variables."

(18) Dr. J. R. CONNER: "Correspondences associated with the rational plane quintic curve."

(19) Professor L. P. EISENHART: Conjugate systems and envelopes of spheres."

(20) Mr. JOSEPH LIPKE: "Natural families of curves in a general curved space of n dimensions."

(21) Professor O. E. GLENN: "On the discriminants of ternary forms."

Mr. Chapman's paper was presented to the Society and read by Professor C. N. Moore. In the absence of the authors, the papers of Professor Snyder, Mr. Hawkesworth, Dr. Miles, Professor Saurel, and Professor Bates were read by title. Professor Bates's paper has appeared in full in the January *Transactions*. Professor Saurel's paper will be published in the BULLETIN. Abstracts of the other papers follow below. The abstracts are numbered to correspond to the titles in the list above.

1. The main result of Mr. Chapman's paper is a proof of the following theorem: If the integral

$$\int_a^\infty f(x)dx$$

is summable (C_r), i. e., if

$$\lim_{x=\infty} \frac{r!}{x^r} \int_a^x \int_a^{\alpha_1} \int_a^{\alpha_2} \cdots \int_a^{\alpha_r} f(\theta) d\theta d\alpha_r \cdots d\alpha_2 d\alpha_1 .$$

exists, and if furthermore $f(x)/x^r$ is uniformly continuous in the interval $x \geqq k > 0$, then $\lim_{x=\infty} f(x)/x^r = 0$.

For the case $r = 1$, this theorem reduces to one due to C. N. Moore,* except that the condition in the latter theorem that $f(x)$ should be uniformly continuous is replaced by the less stringent condition that $f(x)/x$ should have that same property.

2. A collineation group in ordinary three-space, when represented on the Plücker line coordinates, is a linear group in six variables with a quadratic invariant. If a transformation be effected so that the quadratic invariant becomes

$$\sum_{i=1}^{6} x_i^2 = 0,$$

* Cf. *Transactions*, vol. 8 (1907), p. 314.

the group becomes one of orthogonal substitutions, and may be regarded as a group of rotations about a point in six-dimensional space.

Groups of order 2520 and 11520 in three-space correspond to groups leaving invariant regular solids in six-space. The group of order 25920 has not this property, but is shown by Dr. Mitchell to leave invariant a semi-regular solid of 27 vertices, bounded by 27 regular five-dimensional solids of 10 vertices and 72 regular five-dimensional solids of 6 vertices.

3. The (1, 2) correspondence discussed by Professor Snyder is defined by a triply infinite system of quadric surfaces passing through six basis points. The surfaces of this system passing through a seventh point P will also pass through a second point Q. A three-dimensional involution I exists between P and Q. If the parameters in the equation of the quadrics are regarded as plane coordinates in a second space, a (1, 2) point correspondence between the two spaces is established. The surface of branch points in the second space is a general Kummer surface, and the locus of coincident points in the first is the Weddle surface. A number of birational transformations are discussed and the conditions that the basis points are in one, two, $\cdots$, six fold involution are determined.

4. Mr. Hawkesworth presented the three following dimension theorems:

(1) The number of A-dimensional boundaries of an N-dimensional rectangular figure is

$$\frac{N!}{(N-A)!\,A!}\cdot 2^{N-A}.$$

(2) The number of A-dimensional boundaries of an N-dimensional tetrahedroidal figure is

$$\frac{(N+1)!}{(N-A)!\,(A+1)!}.$$

From this it follows that the numerical values of the boundaries must rise and fall symmetrically to a medial value, the number $N+1$ of the vertices equalling that of the last $N-1$ boundaries, the edges and the $N-2$ boundaries being alike $\frac{1}{2}(N+1)\,N$, and the planes and the $N-3$ boundaries each being $(N+1)\,N\,(N-1)/3!$, and so on.

These two formulas hold true, even when A is taken greater than N, the negative and reciprocal result, as for example that there are $\frac{1}{6}$ cubes in a square, meaning simply that, reversely and reciprocally, there are 6 squares in a cube.

(3) In any N dimensional rectilinear figure where N is even the boundaries, taken in their sequence alternately plus and minus (i. e., + vertices − edges + planes − solids + ⋯), always sum up to zero. But if N be an odd number, then the summation gives + 2.

Or, if we count the N-dimensional figure as one of its own boundaries, and thus add − 1 or + 1 to the summation, according as N is odd or even, then the summation is always + 1 for all rectilinear figures. In this case the binomial development of $(2-1)^N$ will give us both the sequence and the summation of the boundaries of a rectangular N-dimensional figure, while in a similar way the symmetrical sequence and summation of those of an N-dimensional tetrahedroidal figure can be represented by $-(1-1)^{N+1}$, omitting the first term of the binomial development.

5. The discriminant of an ordinary differential equation of the second degree, $Ay'^2 + 2By' + C = 0$, is $\Delta = B^2 - AC$. It is supposed that, within the region considered, the coefficients are analytic functions of x and y, and that $y = \eta(x)$ is an analytic branch Γ of the discriminant locus $\Delta = 0$. If the points at which both A and C vanish are excluded, the nature of the solutions in the neighborhood of any point on Γ is known. The object of Professor Longley's paper is an investigation of the integral curves through a point P on Γ at which A, B, and C vanish. It is found that an infinite number of integral curves may pass through P and the nature of these curves is not determined by the nature of Γ. For example, Γ may be a singular solution. Then through every point except P there passes one other integral curve, and it is tangent to Γ; while an infinite number of integral curves may pass through P no one of which is tangent to Γ. Or Γ may be a cusp locus, so that through every point except P there passes one integral curve, and it has a cusp at this point; while an infinite number of integral curves may pass through P no one of which has a cusp at P. In this case it may happen also that for any point except P the integral curve is not tangent to Γ, while at P every one of an infinite number is tangent to P.

6. Klein has considered the problem of determining the two parameters of the differential equation of Lamé in such a manner that the equation furnishes two solutions, one of which vanishes at the end points of a certain interval a_1b_1 and oscillates m times within the interval, and the other vanishes at the end points of a second interval a_2b_2 and oscillates n times in the interval, m and n being positive integers or zero arbitrarily prescribed. Other special problems of a special nature which involve two or more parameters have been treated by Klein and Bôcher.* For the case of two parameters these may be considered as special cases of the following: If $p_i(x) > 0$, $q_i(x) \leqq 0$, $A_{ij}(x), (i = 1, 2; j = 1, 2)$ are given analytic functions of x, when can the parameters λ, μ be so determined that the equations

$$(p_1u')' + q_1u + (\lambda A_{11} + \mu A_{12})u = 0, \quad (p_2v')' + q_2v + (\lambda A_{21} + \mu A_{22})v = 0$$

have solutions $u(x)$, $v(x)$ which satisfy the boundary conditions

$$u(a_1) = u(b_1) = 0, \quad v(a_2) = v(b_2) = 0$$

and oscillate m and n times within the intervals a_1b_1 and a_2b_2 respectively (the intervals a_1b_1 and a_2b_2 may or may not coincide)? By transforming the two equations into a partial differential equation with one parameter, Hilbert showed in a paper read before the Mathematische Gesellschaft zu Göttingen that in case

$$p_1 = p_2 = 1, \; q_1 = q_2 = 0, \; A_{12} = A_{21} = 1, \; A_{12} > 0, \; A_2 < 0$$

there exist an infinite number of parameter pairs $\lambda_1, \mu_1; \lambda_2, \mu_2; \ldots$ for which solutions exist.

Professor Richardson has attacked the problem from the standpoint of the calculus of variations and shows that in case $A_{12} > 0$, $A_{22} < 0$, and $A_{11}(x)\,A_{22}(x) - A_{12}(x)A_{21}(x)$ is not identically zero there exist parameter values λ, μ such that the solutions $u(x)$, $v(x)$ oscillate m and n times respectively. He shows further that the oscillation theorem holds true when the intervals coincide and A_{12} and A_{22} have the same sign throughout the interval while $A_{11}\,A_{22} - A_{21}\,A_{12}$ is different from zero. The results hitherto obtained for problems of this type are special cases of these theorems.

7. The theorem considered in Professor Bôcher's paper states that a homogeneous linear integral equation of the second kind can have only a finite number of linearly independent

*Cf. article by Bôcher, Encyklopädie der mathematischen Wissenschaften, II A 7a.

solutions, and gives an explicit formula bounding this number. The proof depends only on the well known fact that the Gramian of a set of functions is positive or zero according as these functions are linearly independent or dependent.

8. The line-sphere correspondence of Lie owes its importance largely to the fact that it is a contact transformation, intersecting lines corresponding to tangent spheres. In Professor Coolidge's paper it is shown that this is merely a special case of a more general relation connecting distances of lines and angles of spheres. Applications are made to the linear complex and congruence, and the dual projective geometry of two ternary domains. The paper has been published in the January *Transactions*.

9. In Dr. Miles's paper the absolute minimum of a definite integral

$$J = \int_{t_0}^{t_1} F(x, y, x', y')dt$$

in a special field is discussed. Under the assumptions ordinarily imposed on the function F and the admissible curves it is known that there exists a one-parameter family of extremals passing through the point 0. This family has in general an envelope and it is here supposed that the envelope has a cusp. The covering of the region near the cusp point is considered. It is shown that outside the V-shaped cusp region there is a single extremal through each point, while for points within there are three, one of which has already touched the envelope. Then the field about an extremal arc whose end points are conjugate is discussed. The assumption is made that the conjugate point is a cusp point in the envelope, and it is found that the field is a three-sheeted affair similar to a Riemann surface, the sheets being joined by the two branches of the envelope. Further the envelope law holds along either branch of the envelope. A curve M in the V-shaped cusp region is then determined which marks the cessation of absolute minimum. This curve is continuous, passes through the cusp point 1, and is such that the extremal ceases to furnish an absolute minimum as soon as it meets this curve. Finally sufficient conditions for an absolute minimum in this field are considered.

10. In Professor Fields' paper proofs were given of the following theorems:

(1) The degree of a rational function which is adjoint to an algebraic equation for the value $z = \infty$ must be less than the degree of the equation written in the form

$$v^n + F_{n-1}v^{n-1} + \cdots + F_0 = 0,$$

and the degree of the element of the function involving the power v^{n-1} must be $\leqq n - 1$.

(2) In the case of an integral algebraic equation, an integral rational function must be divisible by the factor $z - a$ if its order of coincidence with the n branches corresponding to the value $z = a$ severally exceed by 1 the orders of coincidence defining adjointness for these branches.

(3) If an integral rational function is adjoint for $n - 1$ of the n branches of an integral algebraic equation corresponding to a value $z = a$, it must also be adjoint for the remaining branch.

The principle employed in the proof of each of the above theorems is the same. It is possible in a certain manner to construct rational functions which for a given value of the variable z possess any desired set of adjoint orders of coincidence. In the case of a finite value $z = a$ these special functions are integral. In any case they conform to the requirements of the theorem in question. Now if there exists a function which fails to accord with one of the theorems, the sum obtained on adding to this function one of the corresponding special functions will also fail to accord with the theorem. Proper selection however of the function to be added would, in the case of either of the first two theorems, give a sum whose orders of coincidence with the n branches are indefinitely great, while in the case of the third theorem the orders of coincidence of the corresponding sum with $n - 1$ of the branches would be indefinitely great. Such orders of coincidence are readily shown in each case to be compatible only with the requirements of the theorem in question. This however is in contradiction to our first deduction that the sum of a function which does not accord and a function which does accord with one of the theorems does not itself accord with the theorem. It follows therefore that no function to which any one of the theorems has reference fails to conform to that theorem.

11. Professor Decker's paper establishes a method for the determination of the order of the restricted system of equations which arise from putting equal to zero the determinants obtained

by suppressing each set of $n-m$ columns of a matrix of m rows and n columns. The method of treatment is based on the linear independence of certain of the determinants and the expression of the others in terms of them. The apparent order of the original system is found to depend upon that of a second and this in turn upon that of a third, etc., until finally a reduction formula is established by means of which the determination of the order is made to depend upon the orders of determinants.

The order is calculated for certain cases. These results specialized with regard to the degree of the elements compared with Segre's specialized with regard to the order of the determinants are found to agree. They also confirm the statement of Salmon.

13. The object of Professor Coble's first paper is to estimate the utility of E. H. Moore's cross ratio group for the solution of the sextic. The group appears as the natural bond between the given sextic and a resolvent sextic of the general diagonal type. A material algebraic advantage is gained by using this resolvent rather than the given sextic.

14. In Professor Coble's second paper some formulas of the first are employed to obtain an equation of a cubic surface mapped from a plane by means of cubic curves on a given six-point. The equations of the forty-five triple tangent planes of the surface are derived. An interesting further result is a complete system of invariants for the plane six-point.

15. Two consecutive tangent S_3's to a hypersurface in S_4 will intersect in a plane which is tangent to the hypersurface. This plane and the line joining the points of contact are said to be conjugate. In this paper Professor Moore has discussed this correspondence, also the correspondence between the line of intersection of three consecutive tangent S_3's and the plane determined by the points of contact.

17. In the usual treatment of the Galois theory of equations the property of $x_1, \cdots, x_n$ of being roots of an algebraic equation in one variable is not essential. These quantities may therefore be generalized without affecting the validity of many results in that theory. In the paper of Dr. Phillips they are considered as multipartite numbers representing points in a plane or space of higher dimensions.

18. Making use of the properties of the rational norm-curve in a space of five dimensions, Dr. Conner proves the existence of certain multiple correspondences between the planes of the two curves $\rho x_i = (a_i t)^5$; $\rho \eta_i = (b_i t)^5$ $(i = 0, 1, 2)$, where the b's are three linearly independent forms apolar to the a's. The significance of these correspondences for the general rational plane quintic curve is then discussed.

19. Darboux has considered two parameter families of spheres which possess the property that the lines of curvature on the two sheets of the envelope correspond, and he has shown that the surface of centers of the spheres is cut in a conjugate system by the developables of the congruences of normals to the two sheets. Guichard has called a conjugate system of this sort a "reseau 20." Professor Eisenhart considers the determination of these systems on any surface. It is reducible to the solution of a system of linear partial differential equations of the first order. All conjugate systems with the same Gaussian representation as a "reseau 20" are likewise of this type, and with one of them there is associated a cyclic system whose circles pass through a point. This unique surface may be taken as characteristic of the group. Several interesting particular cases are discussed, one of them leading to results concerning surfaces of Monge.

20. In a paper read at the Princeton summer meeting of the Society, Mr. Lipke gave the complete geometric characterization of natural families of curves on a general surface in euclidean space of three dimensions. Natural families of curves are defined as the systems of extremals connected with variation problems of the type $\int F ds = minimum$, where ds is the element of arc length in the space considered, and F is any point function. It is the purpose of the present paper to give the complete geometric characterization of natural families of curves on a general hypersurface V_n in a euclidean S_{n+1}. The characteristic properties are: (1) the locus of the centers of geodesic curvature of the ∞^{n-1} curves passing through a point is a euclidean S_n; (2) the n directions in which the osculating geodesic circles (circles of constant geodesic curvature) hyperosculate are mutually orthogonal. The natural families are a special type of a much larger class of curves which are characterized by property (1). We find that our two-dimensional results are included in our n-dimensional results.

21. The ordinary discriminant of the ternary quadratic form $a_x^2 + b_x x_3 + c x_3^2$ is expressible as the resultant of the binary forms a_x^2, b_x plus c times the discriminant of a_x^2. Professor Glenn proves the corresponding general theorem for the ternary form of order m, and derives explicit formulas for the $\frac{1}{2}m(m-1)$ ordinary discriminants (conditions that the forms degenerate into distinct linear factors) in terms of the coefficients of the form. One of these discriminants is a linear expression in the resultant of a_x^m and b_x^{m-1} and the discriminant of a_x^m, while the rest are obtained from this one by operating by powers of

$$\Delta_1 = m a_0 \frac{\partial}{\partial b_0} + (m-1) a_1 \frac{\partial}{\partial b_1} + \cdots + a_{m-1} \frac{\partial}{\partial b_{m-1}}$$

and

$$\Delta_2 = m a_m \frac{\partial}{\partial b_{m-1}} + (m-1) a_{m-1} \frac{\partial}{\partial b_{m-2}} + \cdots + a_1 \frac{\partial}{\partial b_0}.$$

He also considers the explicit form of the satellite form of the m-ic and its character as an invariant.

F. N. COLE,
Secretary.

THE WINTER MEETING OF THE CHICAGO SECTION.

THE twenty-eighth regular meeting of the Chicago Section of the American Mathematical Society was held at the University of Minnesota, Minneapolis, Minn., on Wednesday, Thursday, and Friday, December 28–30, 1910, in affiliation with the American Association for the Advancement of Science. Five half-day sessions were held, beginning Wednesday afternoon with a meeting in connection with Section A (mathematics and astronomy) to hear the address of the retiring vice-president, Professor E. W. Brown of Yale University, on "The relations of Jupiter with the asteroids." Other papers of a mathematical character at this session and at its continuation on Thursday morning were by Professor F. R. Moulton of the University by Chicago on "The debt of mathematics to astronomy," and of Dr. W. D. Macmillan of the University of Chicago on "An integrable case in the problem of three bodies." The vice-president of Section A, Professor E. H. Moore of the University of Chicago, presided throughout this session.

The last half-day session, on Friday afternoon, was also a joint meeting with Section A and with Sections B (physics) and D (mechanical engineering) of the American Association, to hear the report of Professor E. V. Huntington of Harvard University, chairman of the committee of twenty appointed at a similar meeting three years ago at Chicago, on "The teaching of mathematics to students of engineering." The work of this committee has extended along two distinct lines, first, the gathering of statistics on this subject from about forty schools where engineering is taught in this country, and secondly, the compilation of a set of syllabi on algebra, geometry, trigonometry, dynamics, and numerical computation. The statistics were turned over to a similar committee working under the International Commission on the teaching of mathematics, and the syllabi formed the basis of Professor Huntington's report at this meeting. They have been printed in preliminary form through the cooperation of the Society for the Promotion of Engineering Education and were presented to all members of this conference for criticisms and suggestions either at the meeting or at any time later to the chairman; for this purpose additional copies may be had on request from the chairman. About forty engineers, physicists and mathematicians attended this meeting and a most interesting and animated discussion was participated in by a large number including Professors E. J. Wilczynski, University of Chicago; A. G. Hall, University of Michigan; E. H. Comstock, School of Mines, University of Minnesota; G. A. Miller, University of Illinois; A. E. Haynes, College of Engineering, University of Minnesota; E. F. Nichols, president of Dartmouth College; T. F. Holgate, Northwestern University; Alexander Ziwet, University of Michigan; E. R. Maurer, College of Engineering, University of Wisconsin; Henry Crew, Northwestern University; H. E. Slaught, University of Chicago; B. L. Newkirk, University of Minnesota; E. F. Chandler, University of North Dakota; J. J. Flather, head of the department of mechanical engineering, University of Minnesota, and L. E. Dickson, University of Chicago, who presided at the meeting. Also numerous criticisms of the report were presented in writing either directly to the chairman or through the secretary of the meeting. Among these were communications from President R. S. Woodward of Washington, and Professor William Kent of New York. In conclusion it was resolved that the members

of this body tender hearty thanks to Professor Huntington for the great interest which he has shown in this work and the untiring service which he has given to it, that they commend the work of the whole committee for the preparation of a report which it is believed must operate for betterment along the lines of the recommendations, and while not prepared to approve in all respects the details, especially in the syllabus on dynamics, as shown by the full and free discussion at this meeting, yet they heartily endorse the spirit of the report and thank the officers of the Society for the Promotion of Engineering Education who have shown their friendly cooperation in offering to publish these syllabi in their official Bulletin for the purpose of drawing out further criticisms and suggestions either in printed papers or in written communications to the chairman of the committee. The further report of this committee is to be presented at the next meeting of the Society for the Promotion of Engineering Education in June, 1911.

The remaining three sessions of the Chicago Section were devoted to the reading of mathematical papers. Professor L. E. Dickson, chairman of the section, presided except when relieved, during the reading of his own papers, by Professor Alexander Ziwet and by Professor A. G. Hall. The total attendance of forty-five at these sessions included the following thirty members of the society:

Professor G. N. Bauer, Professor W. E. Brooke, Professor E. W. Brown, Professor G. L. Brown, Professor W. H. Bussey, Professor E. H. Comstock, Professor H. H. Dalaker, Professor L. E. Dickson, Professor J. F. Downey, Professor H. T. Eddy, Professor Peter Field, Professor A. G. Hall, Professor G. W. Hartwell, Professor T. F. Holgate, Professor E. V. Huntington, Professor Kurt Laves, President E. O. Lovett, Professor G. A. Miller, Dr. W. D. MacMillan, Professor E. H. Moore, Professor F. R. Moulton, Dr. L. I. Neikirk, Professor B. L. Newkirk, Professor W. J. Risley, Professor R. R. Shumway, Professor H. E. Slaught, Dr. H. L. Slobin, Dr. E. H. Taylor, Professor E. J. Wilczynski, Professor Alexander Ziwet.

On Thursday evening about one hundred of the astronomers, engineers, physicists, and mathematicians dined together at the Commercial Club and fulfilled in every respect the desire, expressed at the April meeting of the Chicago Section, for a closer affiliation between these representatives of the American

Association. Professor E. H. Moore acted as toastmaster and impromptu addresses were made by Dean F. C. Shenehon, of the School of Engineering, University of Minnesota, and President C. S. Howe, of Case School of Applied Science, representing the engineers; by President E. F. Nichols, of Dartmouth College, and Professor E. L. Nichols, of Cornell University, representing the physicists; and by Professor E. V. Huntington, of Harvard University, and Professor E. W. Brown, of Yale University, representing the mathematicians.

The election of officers of the Section for the ensuing year resulted as follows: chairman, Professor L. E. Dickson, secretary, Professor H. E. Slaught, third member of the program committee Professor J. B. Shaw. It was voted that the chairman of the Section be requested to deliver an address on the occasion of retiring from his office at the winter meeting in 1911.

The following papers were read at this meeting:

(1) Professor G. A. MILLER: "On the use of the co-sets of a group."

(2) Professor L. E. DICKSON: "Congruencial theory of functions of several variables."

(3) Professor L. E. DICKSON: "Generalization of theorems on linear algebras."

(4) Professor JACOB WESTLUND: "On primitive roots of ideals."

(5) Dr. A. B. FRIZELL: "The problem of defining the set of real numbers."

(6) Professor FLORIAN CAJORI: "A historical note on the Newton-Raphson method of approximations."

(7) Professor E. J. WILCZYNSKI: "One-parameter families and nets of plane curves."

(8) Dr. H. T. BURGESS: "Circular numbers for a plane curve."

(9) Dr. H. T BURGESS: "Rational anharmonic curves upon a quadric."

(10) Dr. H. T. BURGESS: "The applications of matrices to cubic forms."

(11) Professor W. J. RISLEY and Professor W. E. MACDONALD: "Envelopes of one-parameter families of plane curves."

(12) Dr. W. D. MACMILLAN: "A reduction of two power series in many variables to two equivalent polynomials."

(13) Dr. H. F. MACNEISH: "The path of light in a medium homogeneous in concentric spherical layers."

(14) Professor KURT LAVES: "The curves of equal action for elliptical coordinates."

(15) Professor L. C. KARPINSKI: "Robert of Chester's translations of the algebra of Al-Khowarazmi."

(16) Professor L. C. KARPINSKI: "Hindu numerals in the Kitab al Fihrist."

(17) Dr. ARTHUR RANUM: "Ruled surfaces and planed hypersurfaces in four dimensional space."

(18) Dr. L. I. NEIKIRK: "Transformation groups and substitutions of an infinite degree."

(19) Professor J. B. SHAW: "Use of quaternions in differential geometry."

(20) Dr. H. L. SLOBIN: "On plane quintic curves."

(21) Professor F. R. MOULTON: "On the construction of a certain class of periodic solutions of the problem of three bodies."

(22) Professor F. B. WILLIAMS: "Curves on quintic scrolls."

Professor Williams was introduced by Professor G. A. Miller. In the absence of the authors the papers of Professors Cajori, Westlund, Karpinski, Shaw, and Williams, Drs. Frizell, Burgess, MacNeish, and Ranum were read by title. Abstracts of the papers follow below, the numbers corresponding to those in the above list.

1. If H is any subgroup of the group G, the operators $S_2, S_3, \cdots, S_\rho$ may be so selected that every operator of G is found once and only once in each of the following two sets:

$$H + HS_2 + HS_3 + \cdots + HS_\rho,$$
$$H + S_2H + S_3H + \cdots + S_\rho H.$$

The sets HS_α, $S_\alpha H$ ($\alpha = 2, 3, \cdots, \rho$) are called the co-sets of G as regards H, the former being called right co-sets and the latter left co-sets. When H is added to these co-sets they are called respectively right and left augmented co-sets. The operators of G may also be represented uniquely in each of the following two ways, H_1 and H_2 being two arbitrary subgroups of G:

$$H_1H_2 + H_1S_2H_2 + \cdots + H_1S_\lambda H_2,$$
$$H_2H_1 + H_2S_2^{-1}H_1 + \cdots + H_2S_\lambda^{-1}H_1.$$

The sets $H_1S_\alpha H_2$ ($\alpha = 2, 3, \cdots, \lambda$) are called double co-sets of G as regards H_1 and H_2, and when H_1H_2 is added they are called the augmented double co-sets as regards these subgroups.

The objects of Professor Miller's paper are to exhibit how a number of known theorems may readily be proved and related by employing these concepts explicitly, and especially to derive some important relations between abstract groups and substitution groups. The following theorems are proved: (1) If G is represented as a transitive substitution group K, and if K_1 is the subgroup composed of all the substitutions of K which omit a given letter, then any subgroup K' of K has the same number of transitive constituents as the subgroup which corresponds to K_1 has when K is represented with respect to K'. Moreover, the transitive constituents in the two given subgroups have the same relative degrees in the two given representations of G. (2) The product H_1H_2 cannot involve the inverse of each operator in a double co-set with respect to H_1, H_2.

2. The term function is used, for brevity, in Professor Dickson's first paper to designate a rational integral function with integral coefficients. If p is a prime, two functions f and g of a single variable x have, modulo p, a greatest common divisor $d(x)$ which is unique apart from a constant factor and may be determined by an obvious modification of Euclid's algorithm. We have only to multiply the divisor by such a constant that the new leading coefficient is $\equiv 1 \pmod{p}$. The same method does not apply for functions of two or more variables, nor are the resulting theorems entirely analogous to those holding for a single variable. In the latter case $d(x)$ is a linear combination of f and g modulo p. But for $f=(x+1)y-1$, $g=xy+x$ and $p=3$, $xf-(x+1)g=-x(x+2)$, so that the greatest common divisor of f and g is a constant prime to 3; and we do not have $1 \equiv Af+Bg \pmod{3}$, since for $x \equiv 1$, $y \equiv -1$, we have $f \equiv g \equiv 0 \pmod{3}$.

To construct a congruencial theory of functions of two variables (and then by induction for n variables), denote by $d(x)$ the greatest common divisor modulo p of the coefficients $f_i(x)$ in $f=\Sigma f_i(x)y^i$ and call f primitive in y modulo p when $d(x)$ is a constant not divisible by p. We readily show that if fg is divisible modulo p by a function $P(x)$, irreducible modulo p, either f or g is divisible modulo p by $P(x)$. It follows that the product of two functions primitive in y modulo p is primitive in y. If $\phi(x)F$ is divisible modulo p by G, where F and G are primitive in y, then F is divisible by G. A function primitive in y modulo p has as divisors only functions primitive in y.

To establish the existence of a greatest common divisor (f, g) modulo p of f and g, let $g \equiv N \pmod p$, $N = Vy^n + V'y^{n-1} + \cdots$, where V, V', $\cdots$ are functions of x only, and $V \not\equiv 0 \pmod p$. For a suitable integer k, we have $V^k f = Nq_1 + r_1$, algebraically, where r_1 is of degree $< n$ in y. In this way we get

$$V^k f \equiv gq_1 + r_1, \quad V_1^{k_1} g \equiv r_1 q_2 + r_2, \cdots \pmod p.$$

Let r_m be the last remainder not divisible by p. Let $f = \phi(x)F$, $g = \gamma(x)G$, $r_m = \rho(x)R$, where F, G, R are primitive in y modulo p. By use of the preceding lemmas, we readily prove that $R = (F, G), (f, g) = G(\phi, \gamma)$. We may now establish the unique factorization theorem.

The same method is applicable to the theory of algebraic forms; in particular, it avoids the use of Kronecker's theorem concerning the coefficients of the product of two polynomials.

3. The second paper by Professor Dickson deals with linear algebras in which multiplication is not assumed to be associative or commutative. Given two numbers X and Y of the algebra, $X \neq 0$, we can determine a number X' such that $XX' = Y$ or $X'X = Y$ if and only if the right hand or left hand determinant

$$\Delta(X) = \left| \sum_{i=1}^{n} x_i \gamma_{ijk} \right|, \quad \Delta'(X) = \left| \sum_{i=1}^{n} x_i \gamma_{jik} \right|$$

is not zero. Here the γ_{ijk} are the constants of multiplication. We assume that the unit e_1 is a modulus. Then x_1 occurs only in the terms of the main diagonal of Δ or Δ'. Replacing x_1 by $x_1 - \rho$, we obtain the right and left hand characteristic determinants of X

$$\Delta(X - \rho) = \sum_{i=0}^{n} r_i \rho^i, \quad \Delta'(X - \rho) = \sum_{i=0}^{n} l_i \rho^i.$$

Set $X_1 = X$, $X_2 = X^2$, $X_3 = X_2 X$, $X_{i+1} = X_i X$. Similarly, set ${}_{i+1}X = X({}_iX)$. Take $x_1', \cdots, x_n'$ to be the co-factors of the elements in the first row of $\Delta'(X)$. Multiply them by the respective elements of the k-th row of $\Delta'(X)$ and add. We obtain the coefficient y_k of $Y = X'X$. Hence $y_1 = \Delta'$, $y_k = 0\ (k > 1)$. Thus $X'X = \Delta'$. Replace x_1 by $x_1 - \rho$ and let X' become

$$F = \sum_{i=0}^{n-1} F_i \rho^i,$$

where F_i are numbers of the algebra. Thus

$$F(X-\rho)=\sum_{i=0}^{n} l_i\rho^i.$$

By equating coefficients of like powers of ρ, we get

$$F_0X=l_0,\ F_1X-F_0=l_1,\ F_2X-F_1=l_2,\ \cdots,$$
$$F_{n-1}X-F_{n-2}=l_{n-1},\ -F_{n-1}=l_n.$$

Multiply the second on the right by X, the third by X twice, the fourth by X three times, etc. Adding we get

$$\sum_{i=0}^{n} l_iX_i=0.$$

Hence the left-hand characteristic determinant of X vanishes if we replace ρ^i by X_i (i. e., it has X as a right-hand root). Similarly, any element X is a left-hand root of the right-hand characteristic determinant.

Various properties of polynomials in ρ have their analogue in those of the linear combinations of the right hand powers X_i (or the left hand powers of X). If

$$A_0=\sum_{i=0}^{a} a_iX_i,$$

we set

$$A_k=\sum_{i=0}^{a} a_iX_{i+k},$$

whence $A_k=A_{k-1}X$. Then if $\Sigma a_i\rho^i$ and $\Sigma b_i\rho^i$ have the greatest common divisor $\Sigma g_i\rho^i$, we have $A_0=\Sigma m_sG_s$, $B_0=\Sigma n_tG_t$, while $G_0=\Sigma p_sA_s+\Sigma q_tB_t$, in which the m, n, p, q are scalars, $B_0=\Sigma b_iX_i$, $G_0=\Sigma g_iX_i$. In particular, A_0 and B_0 both vanish if and only if G_0 vanishes. Various other results follow in the usual manner.

4. Every prime ideal P in an arbitrary algebraic number field admits, as is well known, of primitive roots. But for P^n, where $n>1$, primitive roots do not always exist. In what cases primitive roots of P^n exist was determined by A. Wiman in 1899.* It is the object of Professor Westlund's paper to treat the somewhat more general problem of actually determining the exponent, mod P^n, to which any primitive root of P belongs.

* Öfversigt af kongliga vetenskapsakademiens Förhandlingar, vol. 56.

5. Studies in the continuum perhaps too often ignore Hilbert's pregnant remark that the totality of real numbers has not yet been defined. Any scheme for producing transfinite series of fundamental sequences is nothing more nor less than a process of defining new sets of irrational numbers. It is a problem of epistemology whether the human intellect can ever exhaust the possibilities of such definition. Dr. Frizell considers the following example.

Let us define inductively a function of $2, 3, \cdots, n$ arguments

$$e(\alpha, \beta) = \alpha^{\beta}, \quad e(\alpha_{i+1}, \alpha_i, \cdots, \alpha_1) = e\{\alpha_{i+1}, e(\alpha_i, \cdots, \alpha_1)\} \quad (i = 1, 2, \cdots, n-1),$$

for example,

$$e(2, 3, 5) = 2^{243}.$$

First let the values of α_i be the well ordered set of positive fractional numbers (integers included, except unity). Next let each α_i take in succession all the above values of the e function and repeat the procedure indefinitely.

Now define a new function f such that for integral values of n $f(\alpha, n) = e(\alpha, \alpha, \cdots, n \text{ arguments})$, e. g., $f(2, 5) = 2^{65536}$. If we succeed in defining $f(\alpha, \beta)$ for all values of α, β in the set defined above by the e function, there will follow the problem of defining in like manner a function f_1 so that $f_1(\alpha, n) = f(\alpha, \alpha, \cdots, (n \text{ arguments}))$, and so on. But we cannot define *all* such functions.

6. In Professor Cajori's paper it is shown that what, since the time of Fourier, has been called "Newton's method" of approximation to the roots of numerical equations is not Newton's method, but Raphson's modification of it. Before 1800 the methods of Newton and Raphson were almost invariably kept distinct. Professor Cajori suggests that the approximation, as described in modern texts, be called the "Newton-Raphson method."

This paper will appear in full in the February number of the *American Mathematical Monthly*.

7. In most treatments of one-parameter families and nets of plane curves, the interest is concentrated altogether upon the determination and discussion of its envelope. More recently some authors, especially Lilienthal and Scheffers, have discussed a number of other interesting problems associated with such

families of curves, but all of these are of a metric nature. In Professor Wilczynski's paper the point of view of projective differential geometry has been adopted, the analytical treatment being based upon the invariant theory of a completely integrable system of partial differential equations of the form

$$y_{uu} = ay_u + by_v + cy,$$
$$y_{uv} = a'y_u + b'y_v + c'y,$$
$$y_{vv} = a''y_u + b''y_v + c''y.$$

The second equation of this system is what Darboux calls a Laplacian equation, and there exists an extensive theory concerned primarily with the cases in which it may be integrated by quadratures. This theory here finds a new and simple interpretation. Moreover, the fact that this equation appears here merely as one of three, instead of by itself, adds deeply to the significance of the theory.

Other points discussed in the paper are the determination of a complete set of invariants and covariants, the osculating conics of the individual curves of the net and their mutual relations.

8. In this paper Dr. Burgess shows that the set of circular numbers defined below are arithmetical invariants of a plane curve under the transformations of the circular group G_6. The small letters are Plücker's well-known numbers.

M: The degree of C is the number of points of C in common with an arbitrary circle. For a circular curve, $M = \mu$.

R: The class of C is the number of circles of an arbitrary pencil touching C. For a circular curve, $R = \nu$.

N: The inflexion of C is the number of osculating circles of C through an arbitrary point. For a circular curve, $N = i$.

A: The number of stationary circles of C. The same for a circular curve.

B: The cuspidal equivalence of C. For a circular curve, $B = \kappa$.

X: The number of doubly orthogonal circles of C through an arbitrary point. For a circular curve, $X =$ the number of binormals.

Y: The number of bitangent circles of C through an arbitrary point. For a circular curve, $Y = \tau$.

G: The number of pairs of osculating circles of C which intersect upon an arbitrary circle. For a circular curve, G is the number of pairs of osculating circles having a common radical axis.

H: The number of pairs of tangent circles of C belonging to an arbitrary net while each circle of the pair passes through the point of contact of the other. For a circular curve all the circles may have their centers upon an arbitrary straight line.

D: The nodal equivalence. For a circular curve, $D + H = \delta$.

I: The number of intersections of C with a minimum line through I, excluding the point I.

J: The number of intersections of C with a minimum line through J, excluding the point J.

F: The number of foci of C, $F = (R - 2I)(R - 2J)$.

9. In his second paper Dr. Burgess proves the following theorem: The necessary and sufficient condition that the anharmonic curves upon the quadric $Q \equiv \sum a_{ik}x_ix_k = 0$, $|a_{ik}| \neq 0$, whose tangents belong to the null system $N \equiv \sum b_{ik}x_iy_k = 0$ may be rational skew curves of degree n, when the matrix $||\lambda a_{ik} + b_{ik}||$ has the characteristic [1111], is that the two positive roots α and β of the equation $|\lambda a_{ik} + b_{ik}| = 0$ be in the ratio $(n-2) : n$.

Proof: The canonical forms in the case [1111] may be taken as $Q \equiv x_1x_4 + x_2x_3 = 0$, $N \equiv \alpha p_{14} + \beta p_{23} = 0$; and Q has the parametric equations $X_1 : X_2 : X_3 : X_4 = uv : u : v : -1$. Determining from these $\rho p_{14} = X_1dx_4 - X_4dx_1 = udv + vdu$ and $\rho p_{23} = X_2dx_3 - x_3dx_2 = udv - vdu$, he finds the differential equation of the curves to be $\alpha(udv + vdu) + \beta(udv - vdu) = 0$. On integrating, the curves are $u = cv^{\frac{\beta+\alpha}{\beta-\alpha}}$, whence in the parameter v the curves are given by

$$\rho X_1 = cv^{\left(\frac{\beta+\alpha}{\beta-\alpha}+1\right)}, \quad \rho X_2 = cv^{\frac{\beta+\alpha}{\beta-\alpha}}, \quad \rho X_3 v, \quad \rho X_4 = -1.$$

Obviously the necessary and sufficient condition that the curves be rational and of degree n is

$$\frac{\beta+\alpha}{\beta-\alpha} + 1 = n, \text{ whence } \frac{\alpha}{\beta} = \frac{n-2}{n}.$$

In like manner, he finds in the other canonical forms that the cases [(11)11], [(11)(11], [(22)] and [(31)] are rational and of degrees 2, 1, 1, and 2 respectively. The one remaining form [22] has the curves

$$X_1 : X_2 : X_3 : X_4 = cue^{2\alpha u} : u : ce^{2\alpha u} : -1,$$

and these are not rational for any value of the root α.

10. In his third paper, Dr. Burgess derives the following method for carrying out a linear substitution with matrix H upon a cubic form $\sum a_{ijk}x_ix_jx_k = 0$ $(i, j, k = 1, 2, \cdots, n)$.

Consider the cubical matrix $\| a_{ijk} \|$ and build from it the series of plane matrices $A_q \equiv \| a_{ijq} \|$ $(q=1, 2, \cdots, n)$. Compute the series of matrix products $H'A_qH$ $(q=1, 2, \cdots, n)$, H' being the conjugate of H. From this series, build the new series $B_p \equiv \| b_{ipk} \|$ $(p=1, 2, \cdots, n)$ by writing for the rth column of B_p the pth column of $H'A_rH$ $(r=1, 2\cdots, n)$. Compute from this last series the set of matrix products $B_pH \| \equiv \| c_{ipk}$ $(p = 1, 2, \cdots, n)$; and construct from these the cubical matrix $\| c_{ijk} \|$. This is the cubical matrix of the transformed cubic form, $\sum C_{ijk}x_i'x_j'x_k' = 0$.

11. Professors Risley and MacDonald discuss the conditions under which a one-parameter family of plane curves will possess an envelope. The discussion is exhaustive for the curves $y = f(x, \alpha)$, and partial for the curves $F(x, y, \alpha) = 0$. The treatment is applicable whether the variables are real or complex.

The authors obtain a necessary and sufficient condition that

$$P(x, y) \equiv y^n + \sum_{i=1}^{n} A_i(x) \cdot y^{n-1}$$

be irreducible at $(x_0, y_0 = 0)$. They then use this theorem and Weierstrass's implicit function theorem to establish a theorem which summarizes and exhausts the facts concerning envelopes of the curves $y = f(x, \alpha)$.

For the curves $F(x, y, \alpha)$, the authors consider two cases: (a) that in which the curves have no multiple points in the region in question, (b) that in which the curves have these multiple points of order two only. They obtain sufficient conditions for an envelope in case (a). Under case (b) they obtain sufficient conditions for the existence of a nodal locus, and conditions necessary and sufficient that a locus of double points be an envelope locus.

12. Let us say that a power series $f(x_1, \cdots, x_s; z_1, \cdots, z_p)$ is of order m in $x_1, \cdots, x_s$ provided the terms of lowest degree in $f(x_1, \cdots, x_s; 0, \cdots, 0)$ are of degree m in $x_1, \cdots, x_s$.

There is a theorem of Weierstrass to the effect that a power series $f(x; z_1, \cdots, z_p)$ of order n in x can be factored in the

form $f(x; z_1, \cdots, z_p) = [x^n + a_1 x^{n-1} + \cdots + a_n] \cdot g(x; z_1, \cdots, z_p)$, where $g(x; z_1, \cdots, z_p)$ is a power series in $x; z_1, \cdots, z_p$ with a constant term distinct from zero, and $a_1, \cdots, a_n$ are power series in $z_1, \cdots, z_p$ vanishing for $z_1 = \cdots = z_p = 0$. Since g does not vanish for $x = z_1 = \cdots = z_p = 0$ it follows that for solutions in the neighborhood of the origin the equations $f = 0$ and $x^n + a_1 x^{n-1} + \cdots + a_n = 0$ are equivalent.

It is the purpose of Dr. MacMillan's paper to show that if $f_1(x, y; z_1, \cdots, z_p)$ and $f_2(x, y; z_1, \cdots, z_p)$ are two power series of order m and n respectively in x and y and if a certain eliminant E is distinct from zero, then two polynomials $p_1(x, y)$ and $p_2(x, y)$ whose coefficients are convergent power series in $z_1, \cdots, z_p$, can be found such that in the neighborhood of the origin the system of equations $p_1 = p_2 = 0$ admits the same solutions for x and y as functions of $z_1, \cdots, z_p$ as does the system $f_1 = f_2 = 0$, and therefore under the hypothesis that $E \neq 0$ the system $f_1 = f_2 = 0$ admits mn solutions for x and y as functions of $z_1, \cdots, z_p$ which vanish for $z_1 = \cdots = z_p = 0$. These results can be used to show that three power series in three dependent variables of order l, m, and n respectively admit in general lmn solutions in the neighborhood of the origin.

13. Dr. MacNeish's paper considers the determination by means of the calculus of variations of the path of light in a medium homogeneous in concentric spherical layers. The question of conjugate points for the various types of paths is especially considered. When the absolute index of refraction n is a continuous function of the position, the path of any ray in the medium is such as to make the integral

$$I = \int_{s_1}^{s_2} n ds,$$

where s is the length of the path, a minimum or a maximum according to the curvature of the refracting surface. If the medium is homogeneous in concentric spherical layers, the path minimizes the integral I. In this case the path lies in a plane through the center of the spheres. The problem is taken up in the plane of the path and six types of paths are obtained, designated types A, B, C, D, E, F, of which only paths of type E may have conjugate points. A path of type E ascends and reaches a maximum and then descends symmetrically to the level of the initial point. If the amplitude of the maximum

point increases with the initial horizontal angle of the path, there are no conjugate points on the path. If the amplitude of the maximum point decreases as the initial horizontal angle of the path increases, the maximum point is a conjugate point for the path.

14. The paper of Professor Laves is a continuation of a former paper published by him in the *Astronomische Nachrichten*. Besides giving a simple geometrical interpretation of Haekel's theorem concerning the integration of Jacobi's partial differential equation for conservative systems, it takes up certain analytical results of Jacobi for elliptical coordinates and shows how they can be simplified by the same geometric conception.

15. The first systematic exposition of algebra was doubtless that written in the early part of the ninth century by Mohammed ibn Musa al-Khowārazmī. Translations from the Arabic into Latin were made in the first half of the twelfth century. One version, probably by Gerhard of Cremona, has been published by Libri in his Histoire des Sciences mathématiques en Italie. An independent version by an Englishman, Robert of Chester — probably also known as Robert of Reading — exists in manuscript in Vienna, in Dresden, and a third copy in the Columbia University library. Professor Karpinski presents a report on the content of this version, showing also that the two Latin versions are doubtless from the same Arabic original of which Frederic Rosen in 1830 published the text and English translation. While several references to the work by Robert of Chester have appeared, no other study of the content has been made.

16. The question of the origin of the Hindu-Arabic numerals is complicated by the fact that two forms of numerals with place value were used by Arabic writers of the tenth and succeeding centuries. Of these the gobār or dust forms resemble more nearly the ones which we use, while the so-called Hindu forms are more like the ones used by the modern Arabs. Sometimes, with both forms, superposed dots were employed instead of zeros to indicate multiples of the powers of ten, as $\ddot{5}$ for 5000. Professor Karpinski has found in the Kitāb al-Fihrist an entirely new and hitherto uncited usage of subscript dots with the same force as the superposed dots.

The reference is of further importance as being one of the earliest (987 A.D.) discussions of the origin of the numerals.

17. Defining a ruled surface in four-dimensional space as a two-dimensional continuum made up of ∞^1 lines, and a planed hypersurface as a three-dimensional continuum made up of ∞^1 planes, Dr. Ranum derives some of the projective properties of these correlative loci. For instance, he shows that every non-developable ruled surface R lies on one and only one developable planed hypersurface P. Given P, an infinite number of R's exist, arranged in pairs of conjugates such that every generator of one meets three consecutive generators of the other; any two conjugates meet in a curve whose tangent surface (developable) also lies on P. Dually, every non-developable planed hypersurface H contains one and only one developable surface S. Given S, an infinite number of H's exist, arranged in pairs of conjugates such that every generating plane of one meets three consecutive generating planes of the other in straight lines; the envelope of the common tangent hyperplanes of any two conjugates is a developable hypersurface, which also contains S.

18. Dr. Neikirk's paper generalizes and extends the results published by him in the *Transactions* for July, 1910. It is shown in the first part that every transformation group defines a substitution group of finite or infinite degree. The problem of constructing the transformation, having given the substitution, is considered and solved in certain cases. The second part is devoted to the discussion of the properties of substitutions of an infinite degree.

19. In Professor Shaw's paper, the advantage of introducing into the analytic treatment of geometry of a surface the differential operator δ, which is a two-dimensional variable Hamiltonian operator analogous to a variable ∇, is shown. The operator δ is defined thus: Let $\rho = \phi(u_1, u_2)$, then $d\rho = \rho_1 du_1 + \rho_2 du_2$. The normal is $\nu = V \cdot \rho_1 \rho_2$. Then

$$\delta = \frac{1}{T\nu}\left(\rho_2 \frac{\partial}{\partial u_1} - \rho_1 \frac{\partial}{\partial u_2}\right).$$

If $\psi(u_1, u_2) = c$ is the equation of a curve on the surface, its tangent is $\delta\psi$. In any direction α tangent to the surface, the differential of ψ is $d_\alpha\psi = TV\alpha\delta\psi$. Evidently $d_\alpha\psi$ is a maxi-

mum for $\alpha \perp \delta\psi$. If ψ is called a "potential" function for the surface, $\psi = c$ gives the lines of equipotential and $V\nu\delta\psi$ the lines of flow.

The differential parameters $\Delta_1(\psi)$, $\Delta_1(\psi, \theta)$, $\Theta(\psi, \theta)$ are

$$\Delta_1(\psi) = -(\delta\psi)^2, \quad \Delta_1(\psi, \theta) = -S\delta\psi\delta\theta, \quad \Theta(\psi, \theta) = TV\delta\psi\delta\theta.$$

The differential parameter $\Delta_2(\psi) = -S\cdot\delta^2\cdot\psi$, where $\delta^2\psi = \delta(\delta\psi)$, remembering that δ is to be differentiated also. The geodetic curvature is $1/\rho_g = S\cdot\delta U\delta\psi$. Around any area on the surface we have

$$\int d\rho\eta = \int\int \delta\cdot q.$$

We may thus have a divergence and a curl.

The method may be at once extended to space of three dimensions considered as lying in one of four; or to one of n dimensions lying in one of $n+1$, or to one of m lying in one of n. The symbolic forms for differential parameters are then seen to be these same formulas in a different notation.

20. In his dissertation Justus Grassman claims to have established the following theorem:

Für die Wendepuncte einer allgemeinen Curve m^{ter} Ordnung gilt der folgende Satz: Legt man durch $\frac{1}{2}(m-2)(m+1)$ Wendepuncte einer Curve m^{ter} Ordnung eine Curve $(m-2)^{ter}$ Ordnung hindurch, so schneidet diese die erstere in $\frac{1}{2}(m-1)(m-2)$ Puncten, die gleichfalls Wendepuncte sind.

It is Dr. Slobin's intention to show that the theorem is not generally true. He applies the theorem to non-singular quintic curves having 15, 11, 7, 3 real inflexions, whose existence has been demonstrated by Barcroft, Klein, and others. With a proper choice of the $\frac{1}{2}(m-2)(m+1)$ inflexions as a base of a sheaf of curves of order $(m-2)$ the theorem leads to a contradiction of Plücker's formulas. We may apply the theorem to curves of higher orders, and it will be found to lead to an impossibility in many cases. The paper also takes up the classification of quintic curves as to their branches, bitangents, and multiple points.

21. This paper treats the case of two finite bodies describing elliptical orbits, and an infinitesimal body moving in the vicinity of the Lagrangian solution points. Professor Moulton discusses the properties of the orbits which are periodic in both two and three dimensions, and shows how to construct the periodic series for their cofficients.

22. The primary object of Professor Williams's paper is to prove, for scrolls of order 5, Professor Story's formula for the number of intersections of any two curves lying on the same ruled surface, i. e.,

$$[C_{a,\alpha}C_{b,\beta}] \equiv a\alpha + b\beta - \mu\alpha\beta,$$

where $C_{a,\alpha}$ is a curve of order a that meets each generator of the ruled surface R_μ of order μ in α points, α being called the grade of the curve, and $C_{b,\beta}$ is a curve of order b and grade β on R_μ. If the curves pass through the same point of a multiple curve on the ruled surface, they are not to be regarded as intersecting there unless they lie on the same sheet. For two curves on R_μ, of multiplicities m and n respectively, say $C_{a=\alpha}$ and $C_{b=\beta}$ the formula is

$$[C_{a=\alpha}C_{b=\beta}] \equiv ma\beta + nb\alpha - \mu\alpha\beta.$$

It is proved that on any scroll R_μ having a director line of multiplicity $\mu - 1$ the number of intersections of $C_{a,\alpha}$ and $C_{b,\beta}$ is equal to the order of the residual intersection of the two cones of orders a and b respectively through these curves from an arbitrary point on the multiple director as a vertex. If S_ν is a surface of order ν, used to cut a curve $C_{b,\beta}$ from the scroll, $\frac{1}{2}\nu(\nu-1)+4$ points for the determination of S_ν can be arbitrarily chosen and S_ν will not break up into R_5 and $S_{\nu-5}$. S_ν must be chosen so that either the residual intersection shall consist entirely of generators or of generators and a curve of order less than b. The following values of ν were found:

$\nu = b$ and S_ν a cone for scroll I (Schwarz's classification, *Crelle*, volume 67).

$\nu = \frac{1}{5}(2b + 8 + n)$, $(n = 0, 1, 2, 3 \text{ or } 4)$ for scrolls III, IV and V.

$\nu = \frac{1}{5}(2b + 11 + n)$, $(n = 1, 2, 3, 4 \text{ or } 5)$ for scrolls II, VI, VII, VIII, IX, X.

$\nu = \frac{1}{5}(2b + 13 + n)$, $(n = 0, 1, 2, 3 \text{ or } 4)$ for scrolls XI and XIV.

$\nu = \frac{1}{5}(2b + 6 + n)$, $(n = 0 \text{ or } 5, 1, 2, 3, 4)$ for scrolls XII and XIII.

And for scroll XV, for $\beta = 1$, $\nu \geqq b - 1$; for $\beta = 2$, $\nu \geqq 2b - 3$; for $\beta \geqq 3$, $\nu \geqq \frac{1}{2}[1 + (2b+1)\beta - 5\beta^2]$.

H. E. SLAUGHT,
Secretary of the Section.

PREPARATION FOR RESEARCH AND THE DOCTOR'S DEGREE IN MATHEMATICS.

*REPORT OF THE AMERICAN SUBCOMMITTEE OF THE INTERNATIONAL COMMISSION ON THE TEACHING OF MATHEMATICS.**

THE following institutions are mentioned in this report:

Brown,† Bryn Mawr, California,† Chicago, Cincinnati, Clark, Columbia, Cornell, Harvard, Illinois, Indiana, Johns Hopkins, Kansas, Michigan, Minnesota, Missouri, Nebraska, Northwestern, Pennsylvania, Princeton, Stanford, Syracuse, Virginia, Wisconsin, Yale.

The requirements for the degree of doctor of philosophy in mathematics are not exactly defined in all the above institutions. As a matter of fact, the policy is in force in some of them of definitely advising aspirants for the doctorate to complete their graduate studies at other universities in which graduate faculties in mathematics have long been strong and active. It is, however, clear that this condition is but temporary. The increase in the number of strong men in mathematics is resulting in added strength in an increasing number of institutions. It is, therefore, certain that the number of universities in which graduate work will be carried to a definite conclusion will be greater in the immediate future than at present.

On the other hand, in those universities in which within the past ten years three or more doctorates have been awarded there is a degree of uniformity in the requirements which probably indicates and establishes a standard for the United States — the outgrowth of conditions peculiar to the educational system of this country. This standard may be described by detailing the course and career of a successful candidate in some one of these institutions.

For admission to registration as a candidate for the doctorate, successful completion of an undergraduate course terminating

* *International Commission on the Teaching of Mathematics.* Committee XIV. — Graduate Work in Universities. Subcommittee 2 — Preparation for Research and the Doctor's Degree. Chairman, P. F. SMITH, Yale University; MAX MASON, University of Wisconsin; E. J. WILCZYNSKI, University of Chicago; J. I. HUTCHINSON, Cornell University; G. D. BIRKHOFF, Princeton University. This report was submitted to the Commissioners in December, 1910.

† Data taken from catalogues and necessarily incomplete.

with a bachelor's degree is required. While all undergraduate degrees cannot at the present time be regarded as equivalent in educational units, greater uniformity in the future will result from the standardizing influence in this direction of the Carnegie Foundation for the Advancement of Teaching.

In his undergraduate years, the candidate has pursued courses in mathematics including the calculus, and in some instances also mechanics, projective geometry, advanced calculus, and theory of functions. In any case the courses mentioned are regarded as introductory, and are required before advanced study begins. A reading knowledge of French and German is prescribed. The course of advanced study upon which the candidate now enters extends usually over three years. This time may, however, be spent in residence at one or more universities, with the exception of the final third year. During this year, the candidate must reside at the university from which he desires his academic degree. The instruction during the period mentioned is for the most part by lectures combined with work in the seminary prepared and delivered by the candidate and his fellow students under the critical eyes of their professors. There may be no formal examinations during this period. The responsibility of mastering his courses rests upon the candidate. In addition to mathematics, one or two minor subjects are required, usually in applied mathematics. This portion of the requirement may, in some cases, be successfully met before the final year.

The mathematical club affords an opportunity for presentation of original papers and discussions of pedagogical and pertinent questions. In the mathematical seminary library are available the standard treatises for study and consultation in connection with the lectures.

The final test of the candidate's power in original investigation is the quality of his thesis. If this is satisfactory, and if he survives the ordeal of a final examination planned to determine the extent of his mathematical attainments, he must, as a final requirement, arrange for the publication of his thesis, in whole or in part, in a manner satisfactory to the department under which he has studied. A small diploma fee is required when the degree is granted.

A summary of the existing conditions at the universities named above follows, the arrangement being topical for the sake of clearness.

Conditions of Matriculation.

The bachelor's degree accepted in matriculation is not specifically described at the following institutions, but all such degrees are considered equivalent: Cincinnati, Illinois, Missouri, Stanford, Syracuse, Virginia, Yale.

"Graduates of this or other institutions in which the bachelor's degree is awarded for courses equivalent to those given at this institution" are accepted at Bryn Mawr, Chicago, Cornell, Kansas, Nebraska, Wisconsin.

A bachelor's degree from a college "in good standing" is required by Brown, California, Clark, Columbia, Johns Hopkins, Princeton, Pennsylvania.

Harvard University requires a bachelor's degree of "substantial character."

At Michigan the credentials of the candidates are examined by the Graduate Council.

It appears that, at the majority of the institutions in question (16 out of 24), the mere attainment of the first degree is not deemed sufficient, but the value of the degree in educational units is decisive.

Language Requirement.

All institutions prescribe a "good reading knowledge" of French and German. The practical point is that the candidate may have to consult treatises and memoirs in French, German, or Italian, and his linguistic knowledge must suffice for this purpose. In addition, Bryn Mawr specifies "some knowledge" of Latin as a requirement, and at Brown and Yale a "good knowledge" of Latin is required unless for some excellent reason the candidate is excused by the faculty.

Time Element.

One year of continuous residence is required in all cases save by Bryn Mawr (2 years), Chicago (1½ years), Columbia (2 years).

Work of equal grade in other universities is generally credited for courses in the early graduate years.

It appears, as a rule, that three years of graduate study are necessary for securing the doctorate.

Prescribed Courses — Number of Courses.

Rigidly prescribed graduate courses in mathematics do not, as a rule, exist. Specific exceptions are the following:

Courses totalling 20 hours for one year at Clark University. The titles of the courses are analytic geometry of higher plane curves, skew curves and surfaces, differential equations and calculus of variations, real and complex function theory including elliptic functions, theory of numbers, modern synthetic geometry, theory of substitutions, invariants, finite differences.

Undergraduate courses amounting to 8 hours for a year at Yale University. The subjects are projective geometry, advanced calculus, mechanics, function theory.

The courses just mentioned are doubtless required at all institutions in which degrees have been awarded, without specific mention of this fact.

The amount of work required is also not generally defined. There are some exceptions. At Yale, a candidate must have successfully followed graduate courses amounting to 18 hours for a year. At Columbia, a minimum of 15 hours is required; at Virginia, 9 hours of lectures each year; at Pennsylvania, 24 year-hours.

The essential point is, of course, that the candidate must prepare himself for the final examination. To make a comparison of the exacting nature of this work in various institutions is difficult and perhaps impossible. The same course (in name) might well be quite vigorously conducted at one university, and at another partake of an easy character. On the other hand, the *breadth* of study required can easily be determined by taking into consideration minor subjects (see the next section).

Minors.

The requirement as to minor subjects at Harvard University is quite definite. From the following four divisions of mathematics: I. Geometry, II. Arithmetic and algebra, III. The differential and integral calculus and the theory of functions, IV. Mechanics and mathematical physics, the candidate must designate one as his chosen field, his major. In each of two of the other three, he must present a thesis on a topic assigned to him. "These minor theses are intended to test the capacity of the candidate to prepare, at short notice, brief expositions of subjects with which he has had previously little or no acquaintance. While no strictly original work is expected on these theses, the form of presentation must be original."

Minor subjects (one or two) are as a rule required, but not specified. Princeton offers an exception in specifying philosophy

and a cognate subject. At Wisconsin, one minor must be studied for two years, a second minor for one year.

At Yale, the requirement of 18 year-hours of graduate courses results in general in the pursuance of courses distributed among three of the main divisions in the Harvard scheme.

Mathematical physics is very generally recommended as a minor.

Final Examination.

Examinations in course are held by Columbia, Harvard, Illinois, Kansas, Missouri, Northwestern, Pennsylvania, Syracuse, Virginia, Wisconsin, Yale.

A formal, final oral examination is required after acceptance of the thesis at the following universities: Chicago, Clark, Columbia, Harvard, Minnesota, Nebraska, Northwestern, Princeton, Wisconsin.

The final examination may be both oral and written at these institutions: Bryn Mawr, California, Cincinnati, Cornell, Illinois, Johns Hopkins, Kansas, Michigan, Missouri, Stanford, Syracuse, Yale.

Yale offers an exception to the general rule of holding a final examination at the end of the candidate's course. At Yale, the examination takes place at the end of the last year but one, or at the beginning of the last year. This determination of the candidate's attainments before work on the thesis is begun is intended as a check.

The final examination is public except at Clark, Cornell, Minnesota, Pennsylvania, and Yale.

Virginia requires no final examination; Pennsylvania, a written examination.

Publication of Thesis.

Publication of the thesis, in whole or in part, is a requirement at all institutions (page 1) except Brown, Harvard, Michigan, Minnesota.

Special conditions imposed are:

There must be deposited in the university library: 150 copies (Bryn Mawr), 25 copies (Northwestern), 100 copies (Clark), 100 copies (Chicago, Cincinnati, Illinois, Nebraska, Virginia, Wisconsin), 50 copies (Cornell), 30 copies (Columbia).

Printing of the entire thesis is required by Bryn Mawr, Chicago, Cincinnati, Columbia, Indiana, Northwestern, Pennsylvania, Princeton, Syracuse, Virginia.

Acceptance by a reputable journal of the whole or part suf-

fices at Clark, Cornell, Johns Hopkins, Indiana, Illinois, Michigan, Missouri, Nebraska, Pennsylvania, Yale.

At Harvard, doctors' dissertations have "almost invariably been published."

Various.

A small diploma fee is usually required. This varies from five dollars to sixty dollars (Princeton).

The master's degree is in no case a prerequisite for the doctorate, but is recommended at Chicago, Cincinnati, Columbia, Illinois, Kansas, Minnesota, Missouri.

Research and Special Courses.

The candidate's ability for original investigation is, in general, stimulated through work in seminaries. Topics assigned to him for private research develop this faculty, and criticism and personal conference with his instructors further the same end. Seminary work consists in discussion and criticism of original papers presented by candidates. Reports on current literature are also called for, in particular on contributions to those special lines in which students and instructors are peculiarly interested.

Equipment—Special Features.

Collections of mathematical models are available, in all institutions, for study by the candidate. At Yale, a special room is equipped with all necessary tools for the making, by the student, of simple models. With the development of graduate clubs at universities throughout the United States has come the establishment in each institution of a mathematical club. At Harvard the club is conducted entirely by the students, members of the faculty attending as guests. The organization at Yale differs from this only in the existence of faculty supervision through the secretary of the club, who is a full professor. This supervision is nominal. In general it may be said that the mathematical club is an informal affair, professors and students meeting for extra-curriculum intercourse and for discussion of such papers and reports as may be presented by faculty and students without obligation so to do.

The library facilities are reported as adequate for all lines of mathematical research, save at Indiana, Kansas, Michigan, Minnesota, Syracuse, and Wisconsin. In addition to the resources of the university library there exists at many institutions a seminary library in which are available current mathematical treatises and periodicals—duplicates of those stored in the main library.

The library of the American Mathematical Society is rapidly becoming for its members a most valuable possession. By exchange, files of all periodicals and proceedings are being completed and these are made available to members wherever located through the regular means of transportation.

Degrees Granted.

Within the years 1900–09, the doctorate in mathematics has been conferred upon approximately 135 candidates, the number at the various institutions being as indicated (in a few cases round numbers are given):

Bryn Mawr 1, California 1, Chicago 30, Clark 9, Columbia 10, Cornell 14, Harvard 12, Illinois 1, Johns Hopkins 20, Kansas 1, Pennsylvania 8, Princeton 3, Stanford 1, Syracuse 1, Virginia 4, Wisconsin 4, Yale 21.

Conditions in the Faculties.

At Clark, Chicago, and Johns Hopkins certain members of the mathematical faculties give graduate instruction only. The number of hours per week of such instruction is respectively: Clark 7, Chicago 6–8, Johns Hopkins 4–7.

In general, graduate instruction is compensated for by a reduction in undergraduate hours. The following table makes clear the policy followed in the various institutions, the first number indicating the hours per week of undergraduate, the second of graduate instruction:

Bryn Mawr	9	and	3
Chicago	6	"	4
Columbia	6	"	6 or 9 and 3
Cornell	11	"	4
Harvard	4½	"	4½ and a reading course
Illinois	10	"	3
Indiana	5	"	4 or 5 and 6
Kansas	7	"	5 or 10 and 3, or 15 and 0
Michigan	12	"	3
Minnesota	11	"	3
Missouri	10	"	2 or 10 and 3
Nebraska	10	"	3 or 15 " 0
Northwestern	6	"	6 or 10 " 3
Pennsylvania	10	"	3
Princeton	6	"	3 or 9 and 3
Stanford	10	"	0
Syracuse	6	"	6
Virginia	12	"	2
Yale	10	"	3

SHORTER NOTICES.

Theories of Parallelism, an historical Critique. By W. B. FRANKLAND. Cambridge, 1910. xviii + 70 pp.

AFTER reading The Story of Euclid (1901), and particularly after reading Euclid Book I with a Commentary (1902), by the same author, every student of the history of mathematics will welcome this little "historical critique," feeling sure of finding much food for thought in small compass. For it is one of the characteristics of Mr. Frankland that he says what he has to say in the fewest words possible, and hence in a book of less than a hundred pages he condenses matter that most writers would expand to fill double the space. And when we come to consider that in these few pages he has presented a scholarly digest of the theories of upward of forty geometers we begin to realize the thought that has been given to the subject and the skill that the author has shown.

Mr. Frankland gives a setting for the historical discussion in an introduction of eighteen pages. In this he begins by stating one existence assumption and one fundamental theorem as follows: "Let us assume that *straight lines* are freely applicable to themselves and to one another; and that there is a *plane* in which they are freely movable; and let us investigate the *parallelism* of such straight lines in an even plane." The theorem is that of Hilbert, that the area of any polygon is proportional to its divergence, that is, to the difference between its angle sum and $(n-2)\pi$. From this theorem, of which the usual proof is given, the author proceeds to prove that for the elliptic, parabolic, and hyperbolic hypotheses, respectively, there are no, one, and two parallels to a given straight line, through any given point.

The historical sketch is then introduced by a discussion of Euclid's own theory, a discussion that has, of course, been anticipated by Dr. (now Sir Thomas) Heath's monumental work on the Elements. Heath calls attention to the fact that Euclid assumes the infinitude of space, and that the possibility of a straight line as re-entrant seems never to have occurred to him. He thus bars from his theory the possibility of the non-existence of parallelism. On the other hand the fifth postulate bars the possibility of double parallelism, so that there is left to him only the parabolic hypothesis. The weakest feature of his theory is that the statement of the fifth postulate positively invites attempts

at proof, and, as we know, these attempts were constantly being made for over two thousand years.

Of the successors of Euclid, Posidonius (B. C. 80) used the equidistant definition, thus barring out the elliptic and hyperbolic hypotheses. Geminus (B. C. 70) has often been made to bear the blame of the definition of one Aganis, that equidistants "are such as lie in one surface, and when produced indefinitely have one space between them, and it is the least line between them." Mr. Frankland follows Dr. Heath in believing this Aganis to have been a writer about A. D. 500. Ptolemy (A. D. 150) wrote a tractate on the fifth postulate, giving four propositions by way of proof. Of these the first is substantially as follows: "If two straight lines are crossed by a transversal so that the interior angles on the same side are together equal to two right angles, then the lines can never intersect." The proof consists in showing that if they meet at O they must also meet on the other side of the transversal at O'. The possibility that O and O' may be identical, as they are in the elliptic hypothesis, evidently did not occur to Ptolemy. Proclus (A. D. 450) seems to have been the first writer to have had any of the modern view of the nature of parallelism. As Mr. Frankland says, his words might almost have been written by Lobachevsky or Bolyai. Without entering into the geometric discussion, one passage is worthy of special note. Proclus says, "It cannot be asserted unconditionally that straight lines produced from less than two right angles do not meet. It is of course obvious that some do meet, but the (euclidean) theory would require all such to intersect. But it may be urged that as the defect from two right angles increases, the straight lines continue asecant up to a certain magnitude of the defect, and for a greater magnitude than this they intersect." It is the last phrase that is significant. The next noteworthy writer on the subject was Nasr-Eddin (A. D. 1250), whose attempt is set forth in the works of Wallis. His effort is well known, tacitly begging the question by another postulate as difficult as that of Euclid. Other writers of more or less prominence are Anaritius (A. D. 900), Gerbert (A. D. 1000), Billingsley, who edited the first English edition of Euclid (1570), Clavius (1574), Oliver (1604), Savile (1621), Tacquet (1654), and Hobbes (1655). In 1663 Wallis gave his proof, or rather his well-known substitute for the postulate. This substitute is assumed "as an universal idea: To any given figure whatever, another figure, similar and of any size, is possible." Leibnitz

(1679) seems to have had "logical premonition of the elliptic hypothesis" in the note that he gives on the definition of Euclid.

Coming to what may be called the modern school, the name of Saccheri (1733) naturally stands out as chronologically the first. His theory is too well known through the work of Engel and Stäckel to need any description in a review, but its prime weakness lay in the failure to consider the possible curvature of three-dimensional space. Simson's (1756) substitute for Euclid's postulate is well known as an educational rather than a mathematical effort. Lambert's (1766), however, is quite the reverse, and essentially he recognizes the three possible hypotheses which Klein finally named the parabolic, elliptic, and hyperbolic. Bertrand's (1778) theory is less familiar, and it has all the charm of style that characterized this writer, but it has not the breadth of view of Lambert's, nor indeed of Saccheri's. Playfair's (1795) adaptation of the proposition of Proclus is well known, since it is the postulate of parallels of our ordinary textbooks, and is noteworthy as an educational measure. The relations of Gauss to the Bolyais are fairly stated, and the hypothesis of the latter is set forth in a clear fashion. To Lobachevsky, however, Mr. Frankland, following Dr. Whitehead and others, gives the greatest praise, both in the matter of priority and of clearness. Such of the modern writers as have contributed to the theory, including Riemann, Cayley, Beltrami, Clifford, and Klein, are mentioned, thus bringing the "critique" up to the present time. Perhaps a quotation from Clifford, that may not have come to the attention of all readers, may be permitted, even at the risk of extending a review already too long: "I hold in fact: (1) That small portions of space are of a nature analogous to little hills on a surface which is on the average flat; namely, that the ordinary laws of geometry are not valid in them. (2) That this property of being curved or distorted is continually passed on from one portion of space to another after the manner of a wave. (3) That this variation of the curvature of space is what really happens in that phenomenon which we call the motion of matter whether ponderable or ethereal. (4) That in the physical world nothing else takes place but this variation, subject, possibly, to the law of continuity." As Frankland remarks, "the boldness of this speculation is surely unexcelled in the history of thought."

DAVID EUGENE SMITH.

Synthetische Theorie der Cliffordschen Parallelen und der linearen Linienörter des elliptischen Raumes. Von WOLFGANG VOGT. Leipzig und Berlin, Teubner, 1909. vii + 58 pp.

THE parallels of the hyperbolic space seemed for many years to furnish the only possible extension of the notion of parallelism derived from euclidean geometry. But about forty years ago the brillant Clifford discovered in elliptic space straight lines which possess most of the properties of euclidean parallels, but differ from them in being skew. Two lines are right (or left) parallel if they cut the same right (or left) generators of the absolute. After Clifford's death, F. Klein and R. S. Ball made extensive contributions to the knowledge of the properties of these lines. More recently E. Study and J. L. Coolidge have been studying the subject. Analytic, synthetic, and differential geometry and vector analysis have been employed by various investigators. But the author of the present paper thinks that no *purely* synthetic treatment has appeared, although such a method is not only possible, but very suitable for the problem. He therefore aims to give the theory of the Clifford parallels in a purely synthetic form.

The article begins with an explanation of the notion of the "winding" of two lines. The Clifford parallels are defined as lines which have more than two common perpendiculars. The nets of parallels of both "windings" are deduced from the v. Staudt-Lüroth theory of line "nets" with imaginary directrices. At the same time, their projective connections with the absolute polar space and with each other are obtained. By elementary methods, it is shown that theorems on euclidean parallels hold true for the Clifford parallels. Other propositions due to modern writers are also demonstrated. The first part of the article is concluded by a brief discussion of the kinematics of elliptic space.

In the second chapter the three sections deal with ruled systems of the second order, the linear complex, and the linear congruence. The projective properties of these linear line loci are assumed and their metrical relations are deduced. Special attention is paid to the appearance of parallels. The parallel cone or the asymptotic cone of a ruled surface of the second order splits, in the elliptic space, into two cones coaxial with the surface. But two arbitrary coaxial cones do not form the parallel cones of a ruled surface of the second order unless certain conditions are fulfilled. A classification of ruled sur-

faces of the second order is made according to the form taken by their cones. It is interesting to compare this with the classifications of quadrics in elliptic space given by J. L. Coolidge (Non-Euclidean Geometry, page 156) and T. J. I'a. Bromwich ("The classification of quadric loci," *Transactions*, volume 6, 1905). In these articles the principles of classification are entirely different from that employed here.

In the section dealing with linear complexes, right and left complexes are distinguished, the existence of "diameter parallel nets" is proved, and the appearance of parallels in the linear complex and in the corresponding null space are investigated. Of special interest is the parallel complex, which possesses a whole net of axes and admits ∞^4 motions carrying it into itself, while the ordinary complex has only ∞^2. The article is concluded by a discussion of the properties of the general linear congruence and some of its special forms.

E. B. COWLEY.

Das Gruppenschema für zufällige Ereignisse. Von HEINRICH BRUNS. Des XXIX Bandes der *Abhandlungen der Mathematisch-Physikalischen Klasse der Königl. Sächischen Gesellschaft der Wissenschaften*, No. VIII. Leipzig, B. G. Teubner, 1906. Pp. 579–628.

THIS monograph is an extension of the brief development of the subject in the eighteenth lecture contained in the treatise by Bruns on Wahrscheinlichkeitsrechnung und Kollektivmasslehre. It is assumed in setting the simplest problem of the work that n balls are drawn, one at a time, from a bag containing balls of various colors, and that each time the ball drawn is returned to the bag before another drawing is made. The n consecutive drawings are called a draw series (Zugreihe) indicated by $Z(n)$. The draw series is written in the form

$$Z(n) = z_1 z_2 \cdots z_n, \tag{1}$$

where z_h denotes the hth drawing.

If the draw series are collected into sets of s with subscripts 1 to s, 2 to $s+1$, 3 to $s+2$, etc., the sets of s are called s-membered draw groups and the symbol $G(s)$ is used to designate such a group. In the formation of such groups, the author distinguishes between what he calls linear and cyclical groups. If, from (1), we take merely z_1 to z_s, z_2 to z_{s+1}, $\cdots$, z_{n-s+1} to z_n,

the formation is called linear, and the corresponding groups linear groups. If, in addition to these groups, we proceed to take sets of s by repeating z's at the left of the draw series (1), the formation is called cyclical and the groups cyclical groups. The larger n is relative to s, the less significant is the difference of the two methods of group formation. It turns out that it is simpler to treat the cyclical groups than the linear groups, and it is of importance that the two do not differ significantly for large values of n.

Associated with each drawing Z_h, the author writes p_h, the probability of drawing that particular color. If in any group $G(s)$ we replace each z by the corresponding p, then, according as the elements p_h occur in $G(s)$, the group takes different forms. To illustrate in a simple case, for three colors 1, 2, 3 the group $G(2)$ takes forms

$$p_1p_1,\ p_1p_2,\ p_1p_3,\ p_2p_1,\ p_2p_2,\ p_2p_3,\ p_3p_1,\ p_3p_2,\ p_3p_3.$$

It is a general analytic representation of the frequency distribution of such forms in a set of drawings that is the first concern of the present work. The simplest case is to determine the frequency with which a form of the group $G(s)$ occurs in each $Z(n)$ where a number of draw series $Z(n)$ have been obtained by drawing say n times. In this case, there is obtained an observed distribution indicated by $U(x)$, where $U(x)$ expresses the relative frequency with which the argument x occurs, and x refers to any group form. Next, the inquiry is for that theoretical distribution $U_1(x)$ of which the observed distribution may be regarded as a sample, or for the distribution that would be obtained if the draw series $Z(n)$ were taken an infinite number of times. The form of the theoretical distribution is given by means of the ϑ-operation introduced in Bruns's treatise.

The problem is next extended in various directions. Instead of confining the inquiry to the distribution of a single group form, several forms are considered simultaneously and different weights are given to the forms considered.

These considerations suggest building an argument

$$x = a_1x_1 + a_2x_2 + \cdots,$$

where a_h is the weight and x_h the frequency of the form of that weight in the totality.

It is shown that by transformations the consideration of one form for purposes of distribution may be replaced by that of one or more other forms, and that it is possible to reduce all the forms until they possess the same number of members. On account of such transformation it is sufficient to keep s fixed in $G(s)$ in treating the problem of distribution.

While the notation is rather complicated, the analytic expression for the frequency of the forms in draw groups seems to be a result fundamental in the theory of "collective quantity" (Kollektivgegenstand) in general, and for problems of statistics in particular, as drawings z_k are representative of any events back of which lies that mode of origination that belongs to problems of chance. The work appears to the reviewer to be of considerable importance for the mathematics of statistics.

H. L. RIETZ.

Elementary Treatise on the Differential Calculus. By W. W. JOHNSON. New York, John Wiley and Sons, 1908. x + 191 pp.

IF phrases current in the present political situation be allowed in reviewing a text in the calculus, the best possible way to describe the impressions made on the reviewer by the present volume would be to say that it is very plainly written from the viewpoint of the "stand-patter" who refuses to be convinced of the value for purposes of instruction in the calculus of the methods of limits and function theory as promulgated by the "progressives," or of the "insurgent," methods of modern disciples of the Perry movement. And the analogy goes further than the stand-pat attitude taken on the method of rates; for it applies throughout to the contents of the 7 chapters of the volume of 191 pages.

The new text is in great part an abridgment of the author's larger treatise on the differential calculus. The contents are very similar to the old, but seemingly compounded in a more digestible form for beginners. The attitude on rates having been taken, the author naturally makes a maximum use of the student's geometric intuition in explaining the fundamental notions of the differential calculus, a point of view sometimes lost sight of by those who, regardless, hold fast to rigor of demonstration.

The derivative, or differential coefficient, is defined as the relative rate of increase of the function as compared with the

independent variable; the absolute rate is obtained when t is taken as the independent variable; and finite values having the rates for their ratio are next assigned to dx and dy; I quote here in part from the preface.

The following chapter headings with the number of pages given to each show the emphasis placed on the several subjects: Functions, derivatives, and differentials, 43; Successive derivatives, 15; Maxima and minima, 17; Evaluation of indeterminate forms, 18; Development of functions in series, 29; Application to plane curves, 52; Functions of two or more variables, 14. From this it will be plainly seen that the evaluation of indeterminate forms and the applications to plane curves certainly receive more attention than is warranted; especially in view of the fact that indeterminate forms to be evaluated by calculus methods arise but rarely in practice and are more or less "cooked up" to suit the occasion. It is doubtful, too, if an elementary calculus is the proper place for a detailed study of the derivation of the equations and properties of the exhaustive list of higher plane curves here studied, even though these have become household words among geometers.

In the abridgment space might well have been saved by omitting entirely the brief references to pedal curves and intrinsic equations. In its place it would be possible to treat more fully such a subject as the radius of curvature, which latter seems almost lost to view.

The problems, with answers, following the several sections are of a type suitable for an elementary text in that they do not seem to be of the kind where the principles of the calculus are lost sight of in the maze of reductions involved in arriving at the answers. If any criticism were to be made, it would surely be to the effect that the answers too often "come out easy" instead of "correct to so many decimal places," a point to be considered since actual problems in the application of the calculus naturally come out in decimals.

ERNEST W. PONZER.

Hyperbolic Functions. Smithsonian mathematical tables. By GEORGE F. BECKER and C. E. VAN ORSTRAND. Washington, Smithsonian Institution, 1909. 8vo. li + 321 pp. $4.50.

THIS volume constitutes the fourth in the set of tables for scientific investigation published by the Smithsonian Institu-

tion, the others being meteorological, geographical, and physical. It is intended to be supplementary to the logarithmic tables in use by all computers. The contents are: Table I, five place values of log sinh u, log cosh u, log tanh u, log coth u; Table II, five place values of the four natural functions; Table III, five place values of sin u, cos u, log sin u, log cos u, the argument u being in radians; Table IV, $\log_{10} e^u$ to seven places; Table V, five place natural logarithms; Table VI, gudermannian of u to seven places in radians and corresponding degrees, minutes, and seconds; Table VII, anti-gudermannian to hundredths of a minute (meridional parts for a spherical globe); Table VIII, radians into degrees. The introduction contains definitions and formulas.

The book is clearly printed and easy of reference. Some of the tables are borrowed, but a large number of entries are new computations either for this collection or for checking old values. The tables cannot fail to be of great service in advancing the use of the hyperbolic functions.

JAMES BYRNIE SHAW.

Problèmes et Exercices de Mathématiques générales. By E. FABRY. Paris, Hermann et Fils, 1910. 8vo. 420 pp. 10 fr.

THE volume on General Mathematics preceding this was reviewed in the BULLETIN, volume 15 (1909), pages 395–399. The present book contains problems and their solutions, to accompany the text proper. The problems occupy 80 pages, the solutions 336. There are 235 problems in algebra, 231 in analytics, 173 in analysis, and 90 in mechanics. These are solved in full in the remaining pages. As a whole the problems would demand considerable ingenuity on the part of the student. For an average student, who knew only what is to be found in the Traité, there are many which would be beyond his range. They are not in this sense exercises. But as a collection of solved illustrations of the subjects treated in the Traité, the book is quite valuable. The student would undoubtedly acquire considerable skill in following these models. This is an excellent storehouse from which extra problems might be drawn for the ambitious students in our American classes.

JAMES BYRNIE SHAW.

Internaciona matematikal Lexiko en Ido, Germana, Angla, Franca, e Italiana. By DR. LOUIS COUTURAT. Jena, Gustav Fischer, 1910. 4to. ii + 36 pp. M. 1.50.

WHETHER a universal language will ever be possible, and whether it will be a living tongue or an artificial construction, remains for the future to show. The fate of Volapük is still fresh in memory, and now the simple language of Esperanto is simplified still more in the new Ido. One may reflect that "le mieux est l'ennemi du bien"; what one considers simple another may not, and as soon as we begin to perfect, we begin to evolve a new thing. To be artificially universal a language must be fixed, crystallized. When one considers the inconsistent notations of mathematics itself and yet their persistence, he may take warning as to the attempt at faultless simplicity.

However, we have here a pamphlet of some thirty-six pages with five columns—Ido—Deutsch—English—Français—Italiano. The first is alphabetic and contains about 1300 words in the artificial tongue Ido. The other columns contain their equivalents. The list follows Müller's Vocabulaire closely, omitting archaic terms and those employed by a single author. The resemblance of the entire list to their French equivalents is rather striking, and one may ask whether after all, it would not be the simplest thing to adopt French as a universal tongue.

JAMES BYRNIE SHAW.

Anfangsgründe der Maxwellschen Theorie verknüpft mit der Elektronentheorie. Von FRANZ RICHARZ. Leipzig, B. G. Teubner, 1909. ix + 245 pp.

THE elements of the theory of the electromagnetic field are now so widely taught, and to so many pupils, that numerous teachers can indulge in the satisfaction of having their own text from which their pupils may get the viewpoint which seems most desirable for them. The literature of the subject therefore grows apace, and as it grows, there devolves upon each successive author the duty to write with some well defined aim rather than to attempt an exhaustive treatment. Apparently Richarz has clearly recognized this duty and privilege and has been successful in their execution. His book must therefore be welcome and advantageous to many teachers who wish to lay the theoretical foundations for the more important applications without being led astray into discussions which at

present are chiefly of but theoretic interest. The author uses the concept of vectors freely, but hardly any vector analysis. Contrary to the growing and now usual custom of employing a right-handed coordinate system for electromagnetism, he uses the left-handed system of the pure mathematician and astronomer.

In Chapter I the start is made directly from the concept of a field and the Maxwell equations are obtained from the Ampère-Faraday experimental laws. The second chapter reviews and expounds and extends the analysis from the point of view of ether and matter. The next chapter treats non-conductors and static phenomena, derives Coulomb's law, discusses free and true electricity and magnetism, all with admirable clearness and precision. Chapter IV brings forward the matter of conductors and conduction currents. Ohm's law and the loss of energy from ether to matter are taken up, but without the introduction of the Poynting vector, which would introduce greater complications than are desired. It should be especially remarked that the concept of a current as a flow of electrons is clearly presented, a thing not often ventured at this stage. A following chapter introduces the potential for conductors and insulators and discusses static or stationary phenomena.

To this point about half the book has been covered. Three long chapters on electromagnetism, induction, and high frequencies complete the work. Proceeding from the general equations of the electromagnetic field, the author introduces the vector potential for a linear conductor and from it he derives the Biot-Savart law for the action between current elements. Stokes's theorem, solid angle as a potential, the multiple value of the potential, and the question of work are mentioned. The mutual potential energy of two circuits and the forces between the circuits are treated with some detail. In much the same way, the work is carried on to induction, two notable features being the derivation of induction in ponderable matter from the point of view of electron theory and the exhibition of the mass of the electron as a phenomenon of self induction. From the fact that such matters are mentioned it appears clear how careful the author is to keep before his readers the latest physical meaning attached to the fundamental phenomena of electromagnetism. High frequency phenomena in dielectrics or conductors are connected with the velocity of light, the opacity of conductors, the coefficient of reflexion, and similar questions in electromagnetic optics.

Such a mere catalog of the important subjects touched upon

by the author may seem hardly a review; it does, however, when taken with the introductory remarks, give perhaps in the shortest space the best idea of a book of this nature where scope and presentation are the two features of most vital interest. The author certainly has written a good introduction to Maxwell's theory and has given appropriate attention to its relation to the later atomic view of electricity.

E. B. WILSON.

NOTES.

THE twenty-ninth regular meeting of the Chicago Section of the AMERICAN MATHEMATICAL SOCIETY will be held at the University of Chicago on Friday and Saturday, April 28–29. Titles and abstracts of papers to be presented at this meeting should be in the hands of the Secretary three weeks in advance of the meeting.

AT the meeting of the London mathematical society held on January 12 the following papers were read: By T. C. LEWIS, "A property of the number 7"; by H. M. MACDONALD, "A mode of representation of an electromagnetic field as due to singularities distributed over the surface"; by W. H. YOUNG, "On the fundamental theorem in the theory of functions of a complex variable"; by E. W. HOBSON, "On the fundamental theorem relating to the Fourier constants for given functions."

THE Paris academy of sciences announces the following prize subjects for the year 1912: The grand prize (3000 francs) for the improvement of the theory of algebraic differential equations of the second or of the third order the general integral of which is uniform. Also the problem mentioned in the BULLETIN, volume 16, page 332. This was proposed before, but no suitable memoirs were received. The Francoeur prize (1000 francs) for work useful to the propagation of pure and applied mathematics. The Poncelet prize (2000 francs) for a worthy memoir in pure mathematics. The Montyon prize (700 francs) for the improvement of machinery useful in agriculture. The Fourneyron prize (1000 francs) for the best contribution to the theory of aviation. A number of prizes in astronomy are announced, without close restriction of the subject.

PROFESSOR L. HEFFTER, of the University of Kiel, has accepted a professorship of mathematics at the University of Freiburg.

DR. F. HARTOGS, of the University of Munich, has been promoted to an associate professorship of mathematics.

PROFESSORS E. JAHNKE, of the mining academy in Berlin, and E. PAPPERITZ, of the mining academy at Freiberg, have been elected to membership in the academy of sciences of Halle.

PROFESSORS J. NEUBERG, of the University of Liège, and P. MANSION, of the University of Ghent, have each been given the rank of professor emeritus. They will retire at the close of the present academic year.

PROFESSOR E. BOREL, of the University of Paris, has been appointed sub-director of the Normal School, as successor to the late Professor J. Tannery.

PROFESSOR M. DULAC, of the University of Algiers, has been appointed professor of mathematics at the University of Lyon.

PROFESSOR E. CARRUS, of the University of Besançon, has accepted the professorship of mathematics at the University of Algiers.

DR. P. BOUTROUX, of the University of Nancy, has been appointed professor of the calculus at the University of Poitiers.

DR. M. FRÉCHET has been promoted to the professorship of rational mechanics at the University of Poitiers.

DR. A. CAHEN will give a course in the theory of numbers at the University of Paris during the second semester of the present academic year.

DR. E. ZORETTI, of the University of Grenoble, will give a course in rational mechanics at the University of Caen.

DR. M. CHAZY has been appointed master of mathematical conferences at the University of Grenoble.

DR. A. CLAEYS, of the University of Ghent, has been promoted to a professorship of mathematics.

DR. M. LAGRULA will assist Professor CARTAN in the newly inaugurated course in general mathematics at the University

of Paris. About 100 students are participating in the exercises, which consist in the evaluation of definite integrals and the use of logarithmic and other numerical tables and of calculating machines.

DR. M. MANDERLI has been appointed docent in mathematics and astronomy at the University of Bern.

AT Princeton University, Professor W. GILLESPIE has been promoted to a full professorship of mathematics.

MR. DUNHAM JACKSON, now studying at Göttingen on a travelling fellowship from Harvard, has been appointed instructor in mathematics at Harvard University for the academic year 1911–1912.

PROFESSORS E. R. HEDRICK, of the University of Missouri, and J. I. HUTCHINSON, of Cornell University, have been granted leave of absence during the second half of the present academic year; they will spend most of their time in Paris. During Professor Hedrick's absence, Professor O. D. KELLOGG will be in charge of the department of mathematics at the University of Missouri.

PROFESSORS E. J. WILCZYNSKI, of the University of Chicago, and C. L. BOUTON and J. L. LOVE, of Harvard University, are spending the present half-year abroad on leaves of absence.

THE marriage is announced of Professor E. D. ROE, JR., of Syracuse University, and Miss JOSEPHINE A. ROBINSON, professor of mathematics at Berea College, at Berea, Ky., February 1.

PROFESSOR G. GUNDELFINGER, of the technical school at Darmstadt, died December 13, 1910, at the age of 65 years.

NEW PUBLICATIONS.

I. HIGHER MATHEMATICS.

ACHITSCH (A.). Eine in der Mechanik anwendbare Näherungs-methode der Auflösung von Differentialgleichungen. (Progr.) Laibach, 1910. 8vo. 17 pp.

AHRENS (W.). Mathematische Spiele. 2te, vermehrte und verbesserte Auflage. (Aus Natur und Geisteswelt, Nr. 170.) Leipzig, Teubner, 1911. 8vo. 6+121 pp. Cloth. M. 1.25

ARTUSBRUNNEN (C.). Die Gleichung für 3 Potenzen gleicher Ordnung mit der Einschränkung des Fermat. Ein Beweis für die Grundregel der Mathematik. Berlin, Buschhardt, 1911. 8vo. 4 pp. M. 2.00

BECKER (K.). Körper grösster Anziehung auf ein und zwei Ellipsoide von n Dimensionen. (Diss.) Halle, 1910.

BENZ (W.). Brennpunktsörter ebener Schnitte einer Flache zweiten Grades. (Progr.) Burgdorf, 1910. 8vo. 91 pp.

BIBLIOGRAPHIE, russische, der Naturwissenschaft und Mathematik. Band IV: 1905. (Russisch.) St. Petersburg, 1910. 8vo. 223 pp. M. 6.00

BÖNINGER (M. H.). Beweis des letzten Fermat'schen Satzes. Frankfurt a. M., Knauer, 1910. 8vo. 16 pp. M. 2.00

BROCKE (E.). Ueber die Benutzung der Beziehungen zwischen Mengen und Zahlen im arithmetisch-algebraischen Unterricht. (Progr.) Zabern, 1909. 4to. 18 pp.

BÜCHER, neue, über Naturwissenschaften und Mathematik. Die Neuigkeiten des deutschen Buchhandels nach Wissenschaften geordnet. Mitgeteilt Herbst 1910. Leipzig, Hinrich. 8vo. pp. 41–63. M. 0.30

DECKER (F. F.). The symmetric function tables of the fifteenthic. Including an historical summary of symmetric functions as relating to symmetric function tables. Washington, Carnegie Institution, 1911. 4to. 16 pp. Paper. $1.25

DIEZ (R.). Die Anfangsgründe der analytischen Geometrie des Raums. Heilbronn, Salzer, 1910. 8vo. 4+160 pp. M. 1.40

DITTRICH (R.). Abstandsörter im Polarraume. (Diss.) Breslau, 1910. 8vo. 119 pp.

DUHEM (P.). Etudes sur Léonard de Vinci, ceux qu'il a lus, ceux qui l'ont lu. Seconde série. Paris, Hermann. 8vo. Fr. 12.00

ENCYCLOPÉDIE des sciences mathématiques pures et appliquées. Tome I, volume 3, fascicule 4: Propositions transcendentes de la théorie des nombres. Exposé par P. Bachmann, J. Hadamard et E. Maillet. Leipzig, Teubner, 1910. 8vo. pp. 47–50 and 289–384. M. 3.00

ERNST (P.). Darstellend-geometrische Behandlung der Dupinschen Zikliden. (Progr.) Wien, 1910. 8vo. 27 pp.

ERNST (P.). I: Die Aout'sche Resultantenkurve. II: Die Axoidale zweier ebener Kurven. (Progr.) Wien, 1909. 8vo. 18 pp.

FEYER (W.). Ueber die Höldersche Funktion und einige verwandte Transcendente. Chemnitz, 1910. 4to. 41 pp. M. 2.00

GIROD, (L.). Das sphärische Analogon der Hypocycloidenbewegung des Cardanus und sein Zusammenhang mit der Theorie eines verallgemeinerten Hooke'schen Gelenkes. (Diss.) Strassburg i. E., 1910.

GOODENOUGH (G. A.). See TOWNSEND (E. J.).

GRELLING (K.). Die philosophischen Grundlagen der Wahrscheinlichkeitsrechnung. Göttingen, Vandenhoeck, 1910. M. 1.40

HAHN (J.). Das sogenannte Schliessungsproblem für zwei Kreise. (Progr.) Dornbirn, 1910. 8vo. 24 pp.

HAMEL (A. B. van). Beginselen van de Differentiaalrekening. Delft, 1910. 8vo. 98 pp. M. 3.20

HANDEL (O.). Einführung in die Differential- und Integralrechnung. Berlin, 1910. 8vo. 122 pp. M. 2.00

HATON DE LA GOUPILLIÈRE. Etude géométrique et dynamique des roulettes planes et sphériques. Paris, Gauthier-Villars, 1910. 4to. 107 pp.

HEATH (T. L.). Diophantus of Alexandria. A study in the theory of Greek algebra. 2nd edition. London, Cambridge University Press, 1910. 12 s. 6d

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HOLTZE (A.). Kleine geometrische Untersuchungen. (Progr.) Naumburg a. S., 1909. 4to. 31 pp.

JAROSCH (J.) Aus der Geometrie der Punkte eines Kegelschnittes. (Progr.) Wien, 1910. 8vo. 18 pp.

KLINGELHÖFFER (R.). Der Fermat'sche Satz. Frankfurt a. M., Knauer, 1910. 8vo. 28 pp. M. 2.00

KOKOTT (P.). Sehnenviereck und elliptische Funktionen. (Progr.) Neisse, 1910. 4to. 9 pp.

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LOURIÉ (S.). Principien der Wahrscheinlichkeitsrechnung. Logische Untersuchung des disjunktiven Urteils. Tübingen, 1910. 8vo. 221 pp. M. 5.80

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PEIRCE (B. O.). A short table of integrals. 2nd revised edition. Boston, Ginn, 1911. 8vo. 144 pp. Cloth. $1.00

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RUSSELL (B.). See WHITEHEAD (A. N.).

SCHAUFLER (A.). Einführung in die perspektive und projektive Geometrie. (Progr.) Heilbronn, 1909. 4to. 30 pp.

SCHMIDT (J.). Der Infinitesimalkalkül. 2ter Teil: Kritik der Infinitesimal-betrachtungen. (Progr.) Wien, 1910. 8vo. 24 pp.

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WHITEHEAD (A. N.) and RUSSELL (B.). Principia mathematica. Volume I. London, Cambridge University Press, 1910. 8vo. 682 pp. 25s

ZACHARIAS (L.). Untersuchungen an zylindrischen Schraubenfedern mit kreisförmigen Querschnitten. (Diss.) Braunschweig, 1910. 8vo. 31 pp.

II. ELEMENTARY MATHEMATICS.

BEMPORAD (A.). Tavole logaritmico-trigonometriche a cinque decimali, nuovamente disposte. Catania, Giannotta, 1910. 8vo. 19 pp. L. 0.80

BERICHTE über den mathematischen Unterricht in Oesterreich. Heft 4: von T. Konradt, J. Loos, und W. Rulf. Wien, Hölder, 1910.

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CAIRO (E.). Sunti di matematica, per la scuola tecnica. (Enciclopedia scolastica diretta da G. M. Gatti, n° 24.) Rocca S. Casciano, Capelli, 1910. 16mo. 88 pp. L. 0.50

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DUSSAUX (E.). Nouveau cours de géométrie. (Collection Julien Boitel.) Paris, Colin, 1910. 18mo. 6+316 pp. Fr. 2.50

EDERT (R.) und KRÖGER (M.). Geometrie für Mittelschulen. Mit besonderer Berücksichtigung der zentrischen und axialen Symmetrie und des geometrischen Zeichnens bearbeitet. 2tes Heft. Veränderte Auflage. Hannover, Meyer, 1911. 8vo. 7 and 113 pp. M. 1.50

FOTH (R.). Anfangsgründe der Zahlen- und Raumgrössen-Lehre. Durchgesehen und berichtigt von A. Müller. 6te Auflage. Hannover, Meyer, 1911. 8vo. 15+303 pp. M. 3.50

FRANCHIS (M. DE). Elementi di geometria. 3a edizione, riveduta, corretta ed ampliata. Palermo, Sandron, 1910. 16mo. 240 pp. L. 2.00

FRINK (F. G.). See HALL (A. G.).

GOTHE (G.). Lehr- und Uebungsbuch der Mathematik für Knaben-Mittelschulen. 1ter Teil: Raumlehre. Leipzig, Freytag, 1911. 8vo. 192 pp. Cloth. M. 2.50

HALL (A. G.) and FRINK (F. G.). Trigonometry. New edition, including spherical trigonometry. (Mathematical series.) New York, Holt, 1910. 8vo. 10+241 pp. Cloth. $1.25

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POIRSON (A.). See LEPOIVRE (G.).

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SCHMIDT (J.). Arithmetik und Geometrie für die Unterstufe der Mittelschulen. 2tes und 3tes Heft. Wien, Hölder, 1910. 8vo. 125+140 pp. M. 3.70

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SCOTO (G.). Nozioni pratiche di geometria. 5a edizione, riveduta e corretta. Palermo, Sandron, 1910. 16mo. 148 pp. L. 1.50

STÄCKEL (P.). Allgemeine Schulen und Fachschulen. Leipzig, Teubner, 1910.

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